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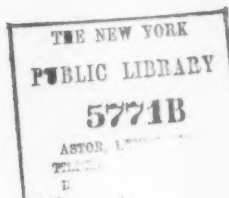
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VAN UVEN'S THEOREM IN PROBABILITY THEORY AND A SELF-RECIPROCAL HANKEL TRANSFORM

S. O. RICE (*New York*)

[Received 10 August 1937]

Introduction

THE solution of the following problem is of interest in some engineering applications of probability theory.

A man starts at the origin of the (x, y) plane and walks a distance l_1 along a line inclined at an angle θ_1 to the x -axis. The angle θ_1 is preassigned and l_1 is a chance variable distributed according to the normal law with standard deviation σ_1 and average value zero. He then turns and walks a distance l_2 along a line inclined at an angle θ_2 to the x -axis, where θ_2 is also preassigned and l_2 is distributed normally about zero with standard deviation σ_2 . What is the chance $p(R)dR$ that after N such walks his distance from the starting-point will lie between R and $R+dR$?

The expression for $p(R)$ is known to be

$$p(R) = 2\rho e^{-\rho^2\sigma} I_0(\rho^2\sigma), \quad (1)$$

$$\text{where } \rho = R/(\sigma^2 - \omega^2)^{\frac{1}{2}}, \quad \sigma = \sum_1^N \sigma_n^2, \quad \omega e^{i\phi} = \sum_1^N \sigma_n^2 e^{2i\theta_n}, \quad (2)$$

ω being obtained by taking the absolute value of the last expression.

It is the purpose of this note to set forth two results which were suggested by a study of this problem. The first one is a restatement of a theorem due to van Uven,* and is connected with the above problem in that it may be used in the derivation of (1). The second result states that $t^{\frac{1}{2}} e^{-bt^2} I_\nu(ct^2)$ is self-reciprocal in the Hankel transformation of order 2ν if

$$4(b^2 - c^2) = 1, \quad R(b - c) > 0, \quad R(b) > 0, \quad R(\nu) > -\frac{1}{2}.$$

This was suggested by comparing (1) with the solution obtained by a method outlined by H. E. Soper.†

Restatement of van Uven's Theorem

Van Uven's statement of the theorem under consideration is substantially as follows.‡ Let u_1, u_2, \dots, u_N be variables such that the

* *K. Akad. v. Wetensch. Amsterdam. Proc.* 16 (1914), 1124-35.

† H. E. Soper, *Frequency Arrays* (Cambridge, 1922), Chap. V.

‡ I have changed the notation somewhat and have used the standard deviation σ_n instead of $h_n = 1/\sigma_n\sqrt{2}$.

probability of u_n lying between v_n and $v_n + dv_n$ is

$$\frac{1}{\sigma_n \sqrt{(2\pi)}} \exp(-v_n^2 / 2\sigma_n^2) dv_n.$$

Let x_1, x_2, \dots, x_M be a set of linear functions of the u 's defined by

$$x_m = \sum_{n=1}^N k_{mn} u_n \quad (m = 1, 2, \dots, M \leq N),$$

and let $a_{mn} = \sigma_n k_{mn}$.

Then the probability that x_j lies between ξ_j and $\xi_j + d\xi_j$ is

$$\frac{(2\pi)^{-M/2}}{\sqrt{B}} \exp\left[-\sum_{i,j=1}^M B_{ij} \xi_i \xi_j / 2B\right] d\xi_1 d\xi_2 \dots d\xi_M, \quad (3)$$

where $B_{ij} = (-)^{i+j} \sum D_i D_j$ and $B = \sum D^2$, while D denotes a determinant of order M taken from the array

$$\begin{array}{cccccc} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1N} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{M1} & a_{M2} & \cdot & \cdot & \cdot & a_{MN} \end{array}$$

and D_i denotes a determinant of order $(M-1)$ taken from the array which is obtained by omitting the i th row in the above array. The two determinants D_i and D_j in the product $D_i D_j$ are to be formed from the same columns of the array.

In the restatement of the theorem the first part extending down to and including expression (3) is unchanged, but the expressions for B and B_{ij} are changed. B is now the determinant of order M whose typical element is b_{ij} , where

$$b_{ij} = \sum_{n=1}^N a_{in} a_{jn} \quad (i, j = 1, 2, \dots, M),$$

and B_{ij} is defined to be the cofactor of b_{ij} in B .

It may readily be shown by the methods used in van Uven's paper that the expressions for B and B_{ij} just given are equivalent to the original ones. In fact, van Uven obtains a result which is only one or two steps removed from the expression for B_{ij} given in the restatement. The essential steps in the transformation may be illustrated by taking the case for $M = 2$ and $N = 3$. Although the outline to be given merely proves a well-known identity, it may perhaps be justified on the ground that it indicates the general method of



procedure. When $M = 2$ and $N = 3$

$$\begin{aligned} \sum D^2 &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}^2 + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}^2 \\ &= - \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ -1 & 0 & 0 & a_{11} & a_{21} \\ 0 & -1 & 0 & a_{12} & a_{22} \\ 0 & 0 & -1 & a_{13} & a_{23} \end{vmatrix}, \end{aligned}$$

which may be verified by developing the determinant by Laplace's method in terms of the two-rowed determinants of the first two rows. The determinant may be put in another form by multiplying the third, fourth, and fifth rows by a_{11} , a_{12} , a_{13} , respectively, and adding them to the first row; then multiplying the same three rows by a_{21} , a_{22} , a_{23} and adding them to the second row. Upon again developing the determinant by Laplace's method we see that it equals $-B$. Thus

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = B = \sum D^2.$$

The original expression for B_{ij} may be shown to be the cofactor of b_{ij} by the same method.

A Self-Reciprocal Hankel Transform

When the problem stated in the introduction is solved by Soper's method, a definite integral is obtained for $p(R)$,

$$p(R) = R \int_0^\infty J_0(Rt) e^{-1/2 t^2} I_0(1/4 t^2) t dt.$$

When the value of $p(R)$ given by (1), which was obtained by a different method, is used, we get an expression for the definite integral on the right. By generalizing this result we are led to the expression

$$\int_0^\infty J_{2\nu}(at) e^{-bt^2} I_\nu(ct^2) t dt = \frac{\sqrt{f}}{a} e^{-bf} I_\nu(cf), \quad (4)$$

where

$$f = a^2/4(b^2 - c^2), \quad \text{and} \quad R(b) > 0, \quad R(b) > R(c), \quad R(\nu) > -\frac{1}{2}.$$

When we restrict b and c by the relation $4(b^2 - c^2) = 1$, we obtain the second result of this note, namely that the function $\sqrt{t} I_\nu(ct^2) e^{-bt^2}$ is self-reciprocal in the Hankel transformation of order 2ν .

In order to prove (4) we assume that b and c are real, $b > 0$, $b > c$, and $R(\nu) > 0$. We use*

$$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi e^{z \cos \theta} \sin^{2\nu} \theta \, d\theta \quad (5)$$

to replace $I_\nu(ct^2)$ in (4). The order of integration may then be changed because of the uniform convergence of the infinite integral. The integration with respect to t may be performed† and the integral on the left in (4) becomes

$$\frac{(\frac{1}{8}ca^2)^\nu}{2\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \frac{\sin^{2\nu} \theta \, d\theta}{(b - c \cos \theta)^{2\nu+1}} \exp \left[-\frac{a^2}{4(b - c \cos \theta)} \right].$$

Changing the variable of integration from θ to ψ where

$$(b^2 - c^2) = (b - c \cos \theta)(b + c \cos \psi)$$

so that
$$\frac{d\theta}{b - c \cos \theta} = \frac{d\psi}{(b^2 - c^2)^{\frac{1}{2}}}, \quad \frac{\sin \theta}{b - c \cos \theta} = \frac{\sin \psi}{(b^2 - c^2)^{\frac{1}{2}}}$$

leads to an integral of the form (5) and hence to the result (4). The restrictions on b , c , ν which were assumed for the purpose of proof may be replaced by those mentioned in connexion with (4) by analytic continuation.

* G. N. Watson, *Theory of Bessel Functions*, 79.

† G. N. Watson, loc. cit., 394.

A NOTE ON A TWO-DIMENSIONAL PROBLEM IN ELECTROSTATICS

By J. HODGKINSON (*Oxford*)

[Received 8 October 1937]

A DISCUSSION of the two-dimensional electrostatic problem—the determination of the forms of the equipotentials when a circular cylindrical dielectric, with one half of its surface covered by a thin conducting layer, is placed in a uniform field—has recently been given by Shepherd who uses a special type of trigonometric series.*

In this note I show how problems, with the conducting layer covering any fraction of the surface of the cylinder, may be solved by means of suitable conformal representations, and I also discuss similar problems with two conducting layers.

Problems with one conducting layer

Take the plane of a cross-section as the plane of the complex variable z . The dielectric occupies the space $|z| \leq 1$, and the position of the conducting layer is given by

$$z = \exp i\theta \quad (-\pi < -\alpha \leq \theta \leq \alpha < \pi).$$

The potential V of an undisturbed uniform field is associated with a function U such that $U + iV = Cz$

where C is a complex constant whose argument determines the direction of the field.

We recall some of the well-known facts about the conformal transformation of two-dimensional electrostatic fields.

Suppose we transform our figure by any relation of the form $z = f(\zeta)$, and let ζ_0 be a pole of $f(\zeta)$. Then

$$U + iV = Cz = Cf(\zeta) = \frac{C'}{\zeta - \zeta_0} + \phi(\zeta),$$

where $\phi(\zeta)$ is analytic at the point ζ_0 . Hence a field which would be uniform except for the presence of bodies in it is transformed into the field due to a doublet at ζ_0 modified by the presence of the corresponding bodies in the transformed field.

It is further obvious that the boundary conditions at the surface of a dielectric are unaltered in character, i.e. if V_i, V_e are the potentials

* W. M. Shepherd, *Proc. London Math. Soc.* (2) 43 (1937), 366–75.

inside and outside the dielectric, $\delta\nu$ is an element of the normal to the surface of the dielectric, and K is the dielectric constant,

$$V_i = V_e \quad \text{and} \quad K \frac{\partial V_i}{\partial \nu} = \frac{\partial V_e}{\partial \nu}$$

at all points on the surface.

The first transformation we make is by means of a linear substitution which transforms the arc $z = \exp i\theta$ ($-\alpha \leq \theta \leq \alpha$) into the positive half and the arc $z = \exp i\theta$ ($\alpha \leq \theta \leq 2\pi - \alpha$) into the negative half of the axis of real numbers in the plane of a variable ζ .

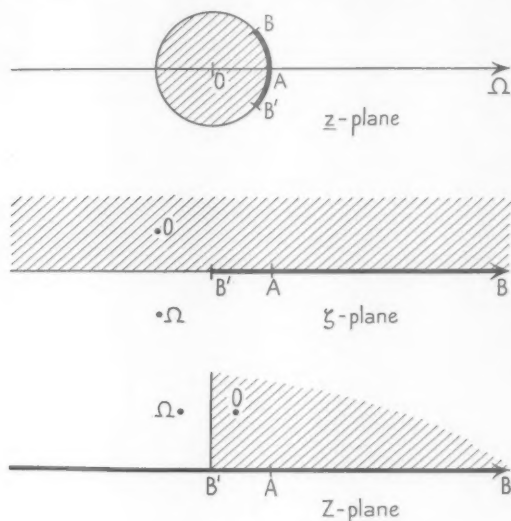


FIG. 1.

The required substitution is

$$\zeta = \frac{-e^{i\alpha}z + 1}{z - e^{i\alpha}}.$$

The points O , Ω become $-e^{-i\alpha}$, $-e^{i\alpha}$, and

$$U + iV \sim \frac{C(e^{i\alpha}\zeta + 1)}{\zeta + e^{i\alpha}} \sim \frac{C(1 - e^{2i\alpha})}{\zeta + e^{i\alpha}} \quad \text{near } \Omega.$$

We next write $\zeta = Z^2$ and take $0 \leq \arg Z \leq \pi$. This substitution transforms the whole ζ -plane slit along the positive half of the axis of real numbers into the upper half of the Z -plane.

The points O , Ω now become $i \exp(-\frac{1}{2}i\alpha)$, $i \exp(\frac{1}{2}i\alpha)$, and

$$\begin{aligned} U+iV &\sim \frac{C(1-e^{2i\alpha})}{Z^2+e^{i\alpha}} \\ &= \frac{C(1-e^{2i\alpha})}{2ie^{\frac{1}{2}i\alpha}} \left\{ \frac{1}{Z-ie^{\frac{1}{2}i\alpha}} - \frac{1}{Z+ie^{\frac{1}{2}i\alpha}} \right\} \\ &\sim \frac{C(1-e^{2i\alpha})}{2ie^{\frac{1}{2}i\alpha}(Z-ie^{\frac{1}{2}i\alpha})} \quad \text{near } \Omega. \end{aligned}$$

Our electrostatic problem is now the determination of the field due to an electric doublet at Ω when the positive quadrant is filled with dielectric and the axis of real numbers is an equipotential.

To this configuration in the Z -plane we adjoin its reflection in the axis of real numbers. Let Ω' be the reflection of Ω . The right-hand half of the plane is now filled with dielectric, and there are equal doublets at Ω , Ω' with axes equally inclined to the axis of real numbers. That axis is clearly an equipotential, and we determine the potentials we require from the potentials in the final configuration.

Changing the notation we write

$$U+iV \sim \frac{\lambda+i\mu}{Z-e^{i\beta}} \quad (\tfrac{1}{2}\pi < \beta < \pi) \quad \text{near } \Omega;$$

i.e., if $Z = X+iY$,

$$V \sim \frac{\mu(X-\cos\beta)-\lambda(Y-\sin\beta)}{(X-\cos\beta)^2+(Y-\sin\beta)^2} \quad \text{near } \Omega.$$

$$\text{Hence} \quad V \sim -\frac{\mu(X-\cos\beta)+\lambda(Y+\sin\beta)}{(X-\cos\beta)^2+(Y+\sin\beta)^2} \quad \text{near } \Omega'.$$

Using the method of images expounded in text-books,* we write

$$\begin{aligned} V_e &= \frac{\mu(X-\cos\beta)-\lambda(Y-\sin\beta)}{(X-\cos\beta)^2+(Y-\sin\beta)^2} - \frac{\mu(X-\cos\beta)+\lambda(Y+\sin\beta)}{(X-\cos\beta)^2+(Y+\sin\beta)^2} \\ &\quad + \frac{\mu'(X+\cos\beta)-\lambda'(Y-\sin\beta)}{(X+\cos\beta)^2+(Y-\sin\beta)^2} - \frac{\mu'(X+\cos\beta)+\lambda'(Y+\sin\beta)}{(X+\cos\beta)^2+(Y+\sin\beta)^2}, \end{aligned}$$

$$V_i = \frac{\mu''(X-\cos\beta)-\lambda''(Y-\sin\beta)}{(X-\cos\beta)^2+(Y-\sin\beta)^2} - \frac{\mu''(X-\cos\beta)+\lambda''(Y+\sin\beta)}{(X-\cos\beta)^2+(Y+\sin\beta)^2},$$

and, on putting in the boundary conditions, we obtain

$$\frac{\lambda}{K+1} = -\frac{\lambda'}{K-1} = \frac{\lambda''}{2}, \quad \frac{\mu}{K+1} = \frac{\mu'}{K-1} = \frac{\mu''}{2}.$$

* e.g. J. H. Jeans, *The Mathematical Theory of Electricity and Magnetism* (5th edition, 1925), 200-1.

The problem is now essentially solved, and I shall not elaborate further details, a matter of routine only.

There is a difference between two-dimensional and three-dimensional electrostatic problems which is often ignored. In three-dimensional problems the convention that the potential vanishes at infinity usually gives a precise meaning to the phrase 'zero potential'. (Even here this precise meaning is lacking when the field contains electrostatic singularities at infinity. For example, when we have a uniform field, it is manifestly absurd to say that any particular plane perpendicular to the direction of the field is the zero-equipotential; nevertheless, it is almost certain that, in the discussion of the problem of a conducting sphere brought into a uniform field, the plane perpendicular to the direction of the field which passes through the centre of the sphere will be selected as the zero-equipotential.) In two-dimensional problems, however, it is quite otherwise. When we write the potential due to a line-charge as $-2e \log r$, we say that the potential vanishes at unit distance from the charge, i.e. the zero-equipotential is determined by the scale with which lengths are measured. An equivalent statement of this is that the zero-equipotential is determined by the arbitrary assignment of the numerical value of the potential at an arbitrarily chosen point. It follows that, in such problems, the words 'earthed' and 'induced charge' have no meaning.

What then is the precise description of the problem we have solved? We note that all the terms in V_e represent doublets. The total charge in the system is therefore nil. We must add to the initial statement of the problem the words 'the conductor continuing to carry no total charge'.

I add here a brief outline of the determination of the field due to a charge on the conducting layer. If the conducting layer carries a charge E per unit length, then $V \sim -2E \log r$ at infinity, where r is the distance from any point in the finite part of the field,

$$\begin{aligned} \text{i.e.} \quad U + iV &\sim -2iE \log z \\ &= -2iE \log \left(\frac{e^{i\alpha}\zeta + 1}{\zeta + e^{i\alpha}} \right) \\ &\sim 2iE \log(\zeta + e^{i\alpha}) = 2iE \log(Z^2 + e^{i\alpha}) \\ &\sim 2iE \log(Z - e^{i\beta}) \quad \text{near } \Omega, \end{aligned}$$

whence $V \sim E \log\{(X - \cos \beta)^2 + (Y - \sin \beta)^2\}$.

Accordingly, we write

$$V_e = E \log \frac{(X - \cos \beta)^2 + (Y - \sin \beta)^2}{(X - \cos \beta)^2 + (Y + \sin \beta)^2} + E' \log \frac{(X + \cos \beta)^2 + (Y - \sin \beta)^2}{(X + \cos \beta)^2 + (Y + \sin \beta)^2},$$

$$V_i = E'' \log \frac{(X - \cos \beta)^2 + (Y - \sin \beta)^2}{(X - \cos \beta)^2 + (Y + \sin \beta)^2},$$

which makes $Y = 0$ an equipotential, and, on putting in the conditions on the boundary of the dielectric, we find that

$$\frac{E}{K+1} = \frac{E'}{-(K-1)} = \frac{E''}{2}.$$

Problems with two conducting layers

When there are two conducting layers, there are three problems to be solved: when (a) the layers carry equal charges of opposite signs, (b) the layers carry equal charges of the same sign, (c) the cylinder is placed in a field with no charge on either layer, the field due to a line-charge being taken as fundamental.

The conducting layers are represented in Fig. 2 by the arcs AB , CD . By a suitable linear transformation the interior of the circle in the z -plane is transformed as before into the upper (shaded) half of a ζ -plane and the exterior into the lower half, with the ζ -point D at infinity and the points A , B , C at e_3 , e_2 , e_1 on the axis of real numbers, where $e_1 + e_2 + e_3 = 0$. Ω is the ζ -point corresponding to the z -point at infinity. We form a two-sheeted surface by adjoining a lower sheet in which the lower half of the sheet is shaded, connexion between the sheets being through AB and CD as branch lines. Ω' , which corresponds to Ω , lies in the upper half of the lower sheet. The lower sheet is, in fact, the reflection of the upper sheet in AB or CD .

We now write $\zeta = \wp(Z)$,

where $\wp'^2(Z) = 4 \prod \{\wp(Z) - e_r\} \quad (r = 1, 2, 3)$

and, in the usual notation, denote the real and purely imaginary periods of the elliptic functions by $2\omega_1$, $2\omega_3$. The Z -plane is divided into rectangles and the strips $2m_1\omega_1 < R(Z) < (2m+1)\omega_1$ are all shaded. The lines $I(Z) = n\omega_3$ are equipotentials. One of the Z -points D is at the origin. Corresponding to the ζ -points Ω , Ω'

we have the Z -points $\alpha \pm i\beta + 2m_1\omega_1 + 2m_3\omega_3$ (lying in the unshaded strips).

The solution of problem (a) is obvious. In the Z -plane configuration we have the problem of the parallel-plate condenser. All conditions are satisfied if we place a uniform distribution 2σ along BA'

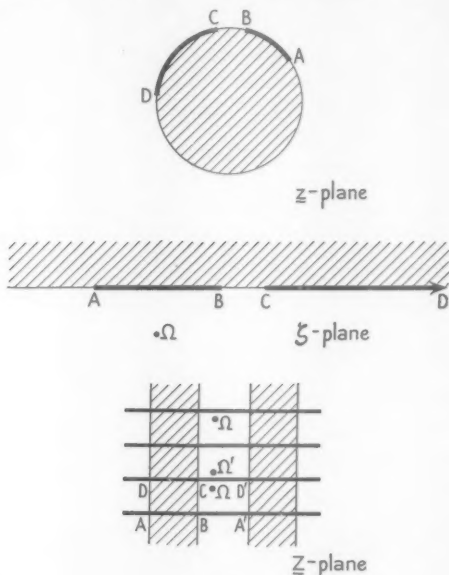


FIG. 2

and all lines congruent to it in the configuration of period parallelograms, $2K\sigma$ along AB and congruent lines, -2σ along CD' and congruent lines, $-2K\sigma$ along DC and congruent lines. In both rectangles $ABCD$, $BA'D'C$ the potential is $-4\pi\sigma Y$. (One half the surface density is spread on each side of the conductor.) The charge on one side of ABA' is $\omega_1(1+K)\sigma$, and the charge on one side of DCD' is $-\omega_1(1+K)\sigma$. By applying the method used in the solution of problem (b) we find that these are the charges per unit length on AB , CD in the original configuration.

To solve problem (b) we first note that, if $2E$ is the total charge per unit length on the two conducting layers in the original configuration, then $U + iV \sim -4iE \log z$ at infinity,

whence

$$U+iV \sim 4iE \log(Z-\alpha-i\beta)$$

near the Z -point $\alpha+i\beta$. Accordingly we give $U+iV$ the correct form of singularity at all the points Ω by writing

$$U+iV \sim 4iE \log \sigma(Z-\alpha-i\beta).$$

The method of images now suggests that we write as trial complex electrostatic potentials*

$$U_e+iV_e = 4iE \log \frac{\sigma(Z-\alpha-i\beta)}{\sigma(Z-\alpha+i\beta)} + 4iE' \log \frac{\sigma(Z+\alpha-i\beta)}{\sigma(Z+\alpha+i\beta)},$$

$$U_i+iV_i = 4iE'' \log \frac{\sigma(Z-\alpha-i\beta)}{\sigma(Z-\alpha+i\beta)}.$$

From these equations we have

$$\begin{aligned} V_e &= 2E \log \frac{\sigma(X+iY-\alpha-i\beta)\sigma(X-iY-\alpha+i\beta)}{\sigma(X+iY-\alpha+i\beta)\sigma(X-iY-\alpha-i\beta)} + \\ &\quad + 2E' \log \frac{\sigma(X+iY+\alpha-i\beta)\sigma(X-iY+\alpha+i\beta)}{\sigma(X+iY+\alpha+i\beta)\sigma(X-iY+\alpha-i\beta)} \\ &= 2E \log \frac{\sigma^2(iY-i\beta)\{\wp(X-\alpha)-\wp(iY-i\beta)\}}{\sigma^2(iY+i\beta)\{\wp(X-\alpha)-\wp(iY+i\beta)\}} + \\ &\quad + 2E' \log \frac{\sigma^2(iY-i\beta)\{\wp(X+\alpha)-\wp(iY-i\beta)\}}{\sigma^2(iY+i\beta)\{\wp(X+\alpha)-\wp(iY+i\beta)\}}, \\ V_i &= 2E'' \log \frac{\sigma^2(iY-i\beta)\{\wp(X-\alpha)-\wp(iY-i\beta)\}}{\sigma^2(iY+i\beta)\{\wp(X-\alpha)-\wp(iY+i\beta)\}}. \end{aligned}$$

It is easily verified that $V_i = V_e$, both when $X = 0$ and when $X = \omega_1$, if $E+E' = E''$.

The second boundary condition is $\partial V_e/\partial X = K \partial V_i/\partial X$ when $X = 0$ and when $X = \omega_1$. Now

$$\begin{aligned} \frac{\partial V_e}{\partial X} &= 2E \left\{ \frac{\wp'(X-\alpha)}{\wp(X-\alpha)-\wp(iY-i\beta)} - \frac{\wp'(X-\alpha)}{\wp(X-\alpha)-\wp(iY+i\beta)} \right\} + \\ &\quad + 2E' \left\{ \frac{\wp'(X+\alpha)}{\wp(X+\alpha)-\wp(iY-i\beta)} - \frac{\wp'(X+\alpha)}{\wp(X+\alpha)-\wp(iY+i\beta)} \right\}, \\ \frac{\partial V_i}{\partial X} &= 2E'' \left\{ \frac{\wp'(X-\alpha)}{\wp(X-\alpha)-\wp(iY-i\beta)} - \frac{\wp'(X-\alpha)}{\wp(X-\alpha)-\wp(iY+i\beta)} \right\}. \end{aligned}$$

* We note that, as we have to satisfy four boundary conditions, viz. $V_i = V_e$ and $K \partial V_i/\partial \nu = \partial V_e/\partial \nu$, along each of BC, DA , the forms we have adopted appear *a priori* inadequate.

Again it may be verified that both boundary conditions are satisfied if $E - E' = KE''$. Hence

$$\frac{E}{K+1} = \frac{E'}{-(K-1)} = \frac{E''}{2}.$$

The charge per unit length carried by CD is found in the usual manner. We describe a circuit about CD in the z -plane in the counter-clockwise direction and suppose the circuit contracted till it is close to CD . Then the total charge is

$$-\frac{1}{4\pi} \int_C^D \frac{\partial V_e}{\partial \nu} ds - \frac{K}{4\pi} \int_D^C \frac{\partial V_i}{\partial \nu} ds,$$

the element $\delta \nu$ of the normal being drawn away from the arc CD . But $\partial V/\partial \nu = -\partial U/\partial s$, so that the charge is

$$-\frac{1}{4\pi} [U_e]_C^D - \frac{K}{4\pi} [U_i]_D^C.$$

In the Z -plane a line just below DCD' corresponds to the circuit round CD in the z -plane.

Now

$$\begin{aligned} U_e = 2iE \log \frac{\sigma(X+iY-\alpha-i\beta)\sigma(X-iY-\alpha-i\beta)}{\sigma(X+iY-\alpha+i\beta)\sigma(X-iY-\alpha+i\beta)} + \\ + 2iE' \log \frac{\sigma(X+iY+\alpha-i\beta)\sigma(X-iY+\alpha-i\beta)}{\sigma(X+iY+\alpha+i\beta)\sigma(X-iY+\alpha+i\beta)}, \end{aligned}$$

so that, just below CD' ,

$$U_e = 2iE \log \frac{\sigma^2(X-\alpha-i\beta)}{\sigma^2(X-\alpha+i\beta)} + 2iE' \log \frac{\sigma^2(X+\alpha-i\beta)}{\sigma^2(X+\alpha+i\beta)}.$$

Hence

$$\begin{aligned} [U_e]_C^{D'} = 2iE \log \frac{\sigma^2(2\omega_1-\alpha-i\beta)\sigma^2(\omega_1-\alpha+i\beta)}{\sigma^2(2\omega_1-\alpha+i\beta)\sigma^2(\omega_1-\alpha-i\beta)} + \\ + 2iE' \log \frac{\sigma^2(2\omega_1+\alpha-i\beta)\sigma^2(\omega_1+\alpha+i\beta)}{\sigma^2(2\omega_1+\alpha+i\beta)\sigma^2(\omega_1+\alpha-i\beta)}, \end{aligned}$$

and this reduces to

$$2i(E-E') \log \frac{\sigma^2(\alpha+i\beta)\sigma^2(\alpha-i\beta-\omega_1)}{\sigma^2(\alpha-i\beta)\sigma^2(\alpha-i\beta-\omega_1)} - 4\pi(pE+qE'),$$

where p, q are undetermined integers. Similarly,

$$[U_i]_D^C = 2iE'' \log \frac{\sigma^2(\alpha+i\beta)\sigma^2(\alpha-i\beta-\omega_1)}{\sigma^2(\alpha-i\beta)\sigma^2(\alpha+i\beta-\omega_1)} - 4\pi rE''.$$

Using the relation $E - E' = KE''$ already established, we find that

$$-\frac{1}{4\pi}[U_c]_C^{D'} - \frac{K}{4\pi}[U_i]_D^C = pE + qE' + KrE''.$$

Since the total charge in the system is $2E$, the charge on AB is $(2-p)E - qE' - KrE''$. Now suppose that the length of the arc AB in the original figure is altered continuously until $AB = CD$. Clearly the conducting layers will now carry charges E per unit length on account of the symmetry of the figure. The Z -figure makes corresponding continuous variations, but the numbers p, q, r being integers must remain fixed. Hence $p = 1, q = 0, r = 0$, i.e. in the original configuration the conducting layers carry equal charges E per unit length.

We have thus verified that the forms V_e, V_i assumed for the external and internal potentials are correct. If it were required to use them for any purposes of calculation, it would be necessary to transform the Weierstrassian functions into Jacobian functions, and subsequently to apply the imaginary transformation to those functions which have imaginary arguments.

Towards the solution of problem (c) we observe that, if we write $\gamma, \delta, -\frac{1}{2}E$ in place of α, β, E , the same argument enables us to write down the external and internal potentials due to a line-charge E per unit length and charges $-\frac{1}{2}E$ per unit length on the two conductors. The only change required concerns the deformation of the figure at the close of the argument. We allow $\gamma + i\delta$ to move continuously in the Z -figure until it lies halfway between the lines ABA', DCD' . This determines the integers p, q, r as before. To the potentials so determined we add the potentials due to equal charges $\frac{1}{2}E$ per unit length on the two conducting layers, and the problem is solved.

ON THE COEFFICIENTS OF SCHLICHT FUNCTIONS

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1. IT is a hitherto unsolved question whether, for every positive integral k , the coefficients of a function, regular and schlicht in $|z| < 1$ and of the form

$$f_k(z) = z + c_{k+1}z^{k+1} + c_{2k+1}z^{2k+1} + \dots$$

do or do not satisfy the inequality

$$|c_n| < A(k)n^{-1+2/k}. \quad (1)$$

It is known that the inequality is true in the cases $k = 1, 2, 3$, and that it is 'nearly' true when $k = 4$, in which case $c_n = O(n^{-1+\epsilon})$.† My present object is to show that the conjecture is certainly not true, nor even 'nearly' true, for all k , and that it does not become true even when we add to the hypotheses the inequality $|f| < A_1$, where A_1 is a certain absolute constant. The related question, whether the n th coefficient of a bounded schlicht function is necessarily of the form $O(n^{-1+\epsilon})$, is incidentally answered at the same time, and in the negative.

Let
$$\phi(z) = \frac{1 + \frac{1}{3}z}{(1 - \frac{1}{3}z)^3} = 1 + \frac{1}{3}z + a_2z^2 + \dots$$

The a_n are positive and absolutely bounded. Let

$$F'(z) = \prod_{m=1}^{\infty} \phi(z^{k^m}),$$

where k is an integer greater than unity, and let $F(z)$ be $\int_0^z F' dz$.

Then we have the following

THEOREM. *When k exceeds a certain absolute constant the function $F(z)$ is regular and schlicht in $|z| < 1$, continuous in $|z| \leq 1$, and bounded by an absolute constant A_1 ; its power series is of the form*

$$F(z) = z + c_{k+1}z^{k+1} + c_{2k+1}z^{2k+1} + \dots;$$

the c_n are non-negative, and the inequality

$$c_n > A(k)n^{-1+o(\log k)}$$

† For the case $k = 1$ see J. E. Littlewood, *Proc. London Math. Soc.* (2), 23 (1925), 481–519 [498]; for $k = 2$, J. E. Littlewood and R. E. A. C. Paley, *J. London Math. Soc.* 7 (1932), 167–9; for $k = 3, 4$, V. Levine, *Math. Zeits.* 38 (1934), 306–11.

holds for an infinite number of values of n , where a is a positive absolute constant and $A(k)$ is a constant depending only on k .†

2. Let A 's denote positive absolute constants, not necessarily the same from one occurrence to another.

It is convenient to consider, along with F , the associated function F^* defined by

$$F^{*'}(z) = \phi(z)F'(z) = \prod_{m=0}^{\infty} \phi(z^{k^m}).$$

When $\rho < 1$ we have evidently $|\phi| \leq e^{A\rho}$, and so

$$|F^{*'}| < \exp\left(A \sum_0^{\infty} \rho^{k^m}\right) < \exp\left(A + \frac{A}{\log k} \log \frac{1}{1-\rho}\right) < A(1-\rho)^{-1},$$

provided k exceeds an appropriate absolute constant, which we suppose.

Remembering that the coefficients of ϕ (and so c_m, c_m^*) are non-negative, we deduce that

$$\sum_1^{\infty} c_m^* = \int_0^1 F^{*'}(\rho) d\rho < A_1. \quad (2)$$

It follows incidentally that F^* is continuous and absolutely bounded (by A_1) in $|z| \leq 1$.

If
$$F_N^{*'} = \prod_{m=0}^N \phi(z^{k^m}),$$

and F_N^*, F_N', F_N have the corresponding meanings, then

$$F^{*'}(z) = F_N^{*'}(z)F^{*'}(z^{k^{N+1}}).$$

Now for any z in $|z| \leq 1$

$$|F^{*'}(z) - F_N^{*'}(z)| \leq F^{*'}(\rho) - F_N^{*'}(\rho) = F^{*'}(\rho)\{1 - F_N^{*'}(\rho)/F^{*'}(\rho)\}.$$

The right-hand side tends to 0 for fixed $\rho < 1$ as $N \rightarrow \infty$, and it is bounded by $A(1-\rho)^{-1}$. Hence the integral of it between $\rho = 0$ and 1 tends to 0, and *a fortiori* F_N^* converges to F^* as $N \rightarrow \infty$, uniformly in $|z| \leq 1$. These various results for F^* will evidently hold also for F , of which incidentally F^* is a majorant.‡

It is easily seen that the coefficients c_n of F have the desired properties. In fact, for n of the form

$$n = 1 + k + k^2 + \dots + k^v,$$

nc_n is the coefficient of $z^{k+k^2+\dots+k^v}$ in

$$(1 + \frac{4}{3}z^k + \dots)(1 + \frac{4}{3}z^{k^2} + \dots)\dots(1 + \frac{4}{3}z^{k^v} + \dots),$$

† The index of n in the inequality is evidently greater than $-1 + 2/k$ when k is large enough.

‡ We say that $\sum C_n z^n$ majorizes $\sum c_n z^n$ if $|c_n| \leq C_n$ for every n .

and so is not less than $(\frac{3}{2})^v > n^{A/\log k}$. It remains only to prove that F is schlicht.

3. In order to prove that F is schlicht it is enough to prove that the (continuous) curve $w = F(e^{i\theta})$ is simple, or that, if α is any value of θ , and $\alpha+t$ another one for which $0 < t \leq \pi$, then

$$F(e^{i\alpha}) \neq F(e^{i(\alpha+t)}). \dagger \quad (3)$$

What we actually prove is that, for k greater than an appropriate absolute constant,

$$|F_N(e^{i(\alpha+t)}) - F_N(e^{i\alpha})|$$

has a positive lower bound independent of N for all large enough N : (3) then follows by making $N \rightarrow \infty$. This naturally requires a good deal of calculation.

We begin by collecting for reference a number of results about ϕ and products of ϕ 's.

Let $\zeta = e^{i\theta}$, and let

$$\phi_n = \phi_n(\theta) = \phi(\zeta^{k^n}), \quad P_n = P_n(\theta) = \prod_{m=1}^n \phi_m(\theta),$$

$$R_n^N = \prod_{m=n+1}^N \phi_m(\theta)$$

all for $n \geq 1$, with the additional conventions \ddagger

$$\phi_0 = 1, \quad P_0 = 1, \quad P_{-1} = 1.$$

LEMMA. For $\phi = \phi(\zeta) = \phi(e^{i\theta})$ we have

$$(a) \quad A < |\phi| < A, \quad |\phi'| < A;$$

$$(b) \quad |\arg \phi| < \frac{1}{2}\pi - A,$$

(a suitable determination of the argument being chosen).

For P_n, R_n^N we have the following results when $n \geq 1$ and $N \geq n$:

$$(c) \quad |P_n| \leq A |P_{n-2}| \leq e^{An};$$

$$(d) \quad \left| \frac{dP_n}{d\theta} \right| < Ak^n |P_{n-2}|, \quad \left| \frac{d}{d\theta} (\phi_{n-1} i e^{i\theta}) \right| < Ak^{n-1};$$

$$(e) \quad \left| \int_{\theta_1}^{\theta_2} (R_n^N - 1) i e^{i\theta} d\theta \right| < \frac{A}{k^{n+1}}, \quad \left| \int_{\theta_1}^{\theta_2} (\phi_n - 1) d\theta \right| < \frac{A}{k^n}$$

for all θ_1, θ_2 .

\dagger If ζ_1, ζ_2 are two distinct values of $e^{i\theta}$, then (to modulus 2π) either $0 < \arg \zeta_2 - \arg \zeta_1 \leq \pi$ or else $0 < \arg \zeta_1 - \arg \zeta_2 \leq \pi$.

\ddagger Note that ϕ_0 does not mean $\phi(\zeta)$ (which does not occur in what follows). Also that when the argument of a function is omitted, or again is θ , we are concerned with a z of the form ζ ; z 's with $|z| < 1$ hardly ever occur again.

If t satisfies the condition

$$(T) \quad \frac{\pi}{k^n} < t \leq \frac{\pi}{k^{n-1}},$$

and $\alpha \leq \theta \leq \alpha + t$, then

$$(f) \quad |P_{n-2}(\theta) - P_{n-2}(\alpha)| < \frac{A}{k} |P_{n-2}(\alpha)|,$$

$$(g) \quad |P_n(\theta)| < A |P_{n-2}(\alpha)|, \quad \left| \frac{dP_n}{d\theta} \right| < A k^n |P_{n-2}(\alpha)|.$$

The inequality $|\phi'| < A$ is trivial. Evidently also

$$|\arg \phi| \leq |\arg(1 + \frac{1}{3}z)| + 3|\arg(1 - \frac{1}{3}z)| \leq 4 \arctan \frac{1}{3} < \frac{1}{2}\pi - A$$

for $|z| \leq 1$. These facts establish (a) and (b). (c) is immediate. For (d) we observe in the first place that

$$\left| \frac{d\phi_m}{d\theta} \right| = |k^m \phi'(\zeta^{k^m})| \leq A k^m \quad (m \geq 1),$$

which gives the second part for $n > 1$, and the case $n = 1$ is trivial. Secondly, we have

$$\left| \frac{1}{P_n} \frac{dP_n}{d\theta} \right| = \left| \sum_1^n \frac{1}{\phi_m} \frac{d\phi_m}{d\theta} \right| < A \sum_1^n \left| \frac{d\phi_m}{d\theta} \right| < A \sum_1^n k^m < A k^n,$$

and the first part follows with the help of (c).

Next, $R_n^N(z) - 1$ is majorized by

$$\prod_{n+1}^{\infty} \phi(z^{k^m}) - 1 = F^{*'}(z^{k^{n+1}}) - 1 = \sum_{m=1}^{\infty} m c_m^* z^{m k^{n+1}-1},$$

whence, integrating term by term,

$$\begin{aligned} \left| \int_{\theta_1}^{\theta_2} (R_n^N - 1) i e^{i\theta} d\theta \right| &\leq \sum_1^{\infty} m c_m^* \left| \int_{\theta_1}^{\theta_2} \zeta^{m k^{n+1}-1} i e^{i\theta} d\theta \right| \\ &\leq \sum_1^{\infty} m c_m^* \frac{2}{m k^{n+1}} < \frac{A}{k^{n+1}}, \end{aligned}$$

by (2). This is the first result of (e); the second is similar and easier.

Finally, we have for $n \geq 1$, t subject to (T), and $\alpha \leq \theta \leq \alpha + t$,

$$\left| \log \left(\frac{P_{n-2}(\theta)}{P_{n-2}(\alpha)} \right) \right| \leq \int_{\alpha}^{\theta} \left| \frac{1}{P_{n-2}} \frac{dP_{n-2}}{d\theta} \right| d\theta \leq A k^{n-2} (\theta - \alpha) < \frac{A}{k}.$$

This gives (f); $|P_{n-2}(\theta)| < A |P_{n-2}(\alpha)|$ is true *a fortiori*; and (g) follows by combination with earlier results.

4. We come now to the main proof. Given α and $\alpha+t$, where $0 < t \leq \pi$, let n be the positive integer for which t satisfies the inequalities

$$(T) \quad \frac{\pi}{k^n} < t \leq \frac{\pi}{k^{n-1}}.$$

There are two genuinely distinct cases, $n = 1$ and $n > 1$, and the latter has a sub-case $n = 2$ which is exceptional in formal respects. We keep the three cases together as long as possible; the conventions are designed to this end.

When $n \geq 1$ (and $N > n$) we have

$$\begin{aligned} F'_N(\zeta) &= \phi_{n-1} \phi_n P_{n-2} R_n^N \\ &= P_{n-2}(\alpha) \phi_{n-1} \phi_n + P_n(R_n^N - 1) + \phi_{n-1} \phi_n \{P_{n-2}(\theta) - P_{n-2}(\alpha)\}, \end{aligned}$$

suppressed arguments being always θ . (The conventions are involved when $n = 1$ and $n = 2$.) Hence, writing ΔF_N for $F_N(e^{i(\alpha+t)}) - F_N(e^{i\alpha})$, we have

$$\Delta F_N - P_{n-2}(\alpha) \int_{\alpha}^{\alpha+t} \phi_{n-1} \phi_n i e^{i\theta} d\theta = T_1 + T_2, \quad (4)$$

where

$$T_1 = \int_{\alpha}^{\alpha+t} P_n(R_n^N - 1) i e^{i\theta} d\theta, \quad T_2 = \int_{\alpha}^{\alpha+t} \phi_{n-1} \phi_n \{P_{n-2} - P_{n-2}(\alpha)\} i e^{i\theta} d\theta.$$

For T_1 we integrate by parts, getting

$$T_1 = P_n(\alpha+t) \int_{\alpha}^{\alpha+t} (R_n^N - 1) i e^{i\theta} d\theta - \int_{\alpha}^{\alpha+t} \frac{dP_n}{d\theta} \left(\int_{\alpha}^{\theta} (R_n^N - 1) i e^{i\theta} d\theta \right) d\theta.$$

Hence we have, by (e) and (g) of the Lemma,

$$\begin{aligned} |T_1| &< A |P_{n-2}(\alpha)| \frac{1}{k^{n+1}} + \int_{\alpha}^{\alpha+t} A k^n |P_{n-2}(\alpha)| \frac{A}{k^{n+1}} d\theta \\ &= |P_{n-2}(\alpha)| \left(\frac{A}{k^{n+1}} + \frac{At}{k} \right) < |P_{n-2}(\alpha)| \frac{At}{k}, \end{aligned}$$

in virtue of (T).

Next, part (f) of the Lemma (together with $|\phi_m| < A$) shows that T_2 satisfies the same inequality as T_1 ; hence, from (4),

$$|\Delta F_N - P_{n-2}(\alpha)J| < |P_{n-2}(\alpha)| \frac{At}{k}, \quad (5)$$

where

$$J = \int_{\alpha}^{\alpha+t} \phi_{n-1} \phi_n i e^{i\theta} d\theta$$

and t is subject to (T).

5. We prove next that, for k greater than an appropriate absolute constant and t subject to (T),

$$|J| > At. \quad (6)$$

We separate the cases

$$(i) \quad \frac{\pi}{k^n} < t \leq \frac{\pi}{k^{n-\frac{1}{2}}},$$

$$(ii) \quad \frac{\pi}{k^{n-\frac{1}{2}}} < t \leq \frac{\pi}{k^{n-1}}.$$

Consider first case (i). When $n > 1$ the argument $k^{n-1}\theta$ of $\zeta^{k^{n-1}}$ in ϕ_{n-1} differs, in the range $\alpha \leq \theta \leq \alpha+t$, from its value when $\theta = \alpha$ by less than $Ak^{-\frac{1}{2}}$; hence (since $|\phi'(z)| < A$)

$$|\arg \phi_{n-1}(\theta) - \arg \phi_{n-1}(\alpha)| < Ak^{-\frac{1}{2}}.$$

This is true (and trivial) also when $n = 1$. Since further

$$|\arg \phi| < \frac{1}{2}\pi - A$$

and

$$|\theta - \alpha| < Ak^{-n+\frac{1}{2}} \leq Ak^{-\frac{1}{2}},$$

it follows that $\arg(\phi_{n-1}\phi_n i e^{i\theta})$ ($\arg \psi$, let us say) differs from the constant $\beta = \arg(\phi_{n-1}(\alpha) i e^{i\alpha})$ by less than $\frac{1}{2}\pi - A$, provided k exceeds an appropriate absolute constant. Since $R(\psi e^{-i\beta}) > A|\psi|$, the modulus of the integral J exceeds A times the integral of the modulus;

$$|J| > A \int_{\alpha}^{\alpha+t} |\phi_{n-1} \phi_n i e^{i\theta}| d\theta \\ > At,$$

by (a) of the Lemma.

6. Consider now case (ii). We have

$$\left| J - \int_{\alpha}^{\alpha+t} \phi_{n-1} i e^{i\theta} d\theta \right| = \left| \int_{\alpha}^{\alpha+t} \phi_{n-1} i e^{i\theta} (\phi_n - 1) d\theta \right| \\ = \left| \left[\phi_{n-1} i e^{i\theta} \int_{\alpha}^{\theta} (\phi_n - 1) d\theta \right]_{\alpha}^{\alpha+t} - \int_{\alpha}^{\alpha+t} \frac{d}{d\theta} (\phi_{n-1} i e^{i\theta}) \left(\int_{\alpha}^{\theta} \right) d\theta \right| \\ < \frac{A}{k^n} + \int_{\alpha}^{\alpha+t} A k^{n-1} \frac{A}{k^n} d\theta,$$

by (e) and (d) of the Lemma,

$$< \frac{At}{k^{\frac{1}{2}}} + \frac{At}{k} < \frac{At}{k^{\frac{1}{2}}}, \quad (7)$$

in virtue of (ii).

If now $n > 1$, then $|\theta - \alpha| < A/k$ in $\alpha \leq \theta \leq \alpha + t$; and so (by an argument we have met already in § 5), $\arg(\phi_{n-1}ie^{i\theta})$ differs from $\arg(ie^{i\alpha})$ by less than $\frac{1}{2}\pi - A$, and

$$\left| \int_{\alpha}^{\alpha+t} \phi_{n-1}ie^{i\theta} d\theta \right| > A \int_{\alpha}^{\alpha+t} |\phi_{n-1}ie^{i\theta}| d\theta > At. \quad (8)$$

This inequality is still true when $n = 1$, since the left-hand side is then

$$\left| \int_{\alpha}^{\alpha+t} ie^{i\theta} d\theta \right| = |e^{it} - 1| > At$$

(since $0 < t \leq \pi$).

7. From (7) and (8) we have

$$|J| > At$$

in case (ii), and hence always. Combining this with (4) and (5), we have finally, when k exceeds a certain absolute constant,

$$|\Delta F_N| > \left(|J| - \frac{At}{k} \right) |P_{n-2}(\alpha)| > At |P_{n-2}(\alpha)|.$$

The right-hand side being positive and independent of N , the proof is completed.

SOLUTIONS OF OSEEN'S EXTENDED EQUATIONS FOR CIRCULAR AND ELLIPTIC CYLINDERS AND A FLAT PLATE

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1. Introduction

THE problem dealt with is that of finding the motion of an incompressible viscous fluid, extending to infinity in all directions, due to the uniform transverse motion of an infinite cylinder. It is convenient to give the whole system a uniform translatory motion so as to reduce the cylinder to rest. It will be assumed that, relative to axes fixed in the cylinder, the motion is steady, and takes place in conformity with the equations

$$\left. \begin{aligned} -v\omega &= -\frac{\partial P}{\partial x} - v\frac{\partial \omega}{\partial y} \\ u\omega &= -\frac{\partial P}{\partial y} + v\frac{\partial \omega}{\partial x} \end{aligned} \right\} \quad (1.1)$$

where, with the usual notation,

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad P = p/\rho + \frac{1}{2}(u^2 + v^2) + \Omega, \quad (1.2)$$

Ω being the potential of the body forces.

Oseen's* modification consists in writing for u and v on the right-hand side of equation (1.1) their constant values at an infinite distance from the cylinder, and neglecting squares of small differences in the formula (1.2) for P . These values will be the velocity components of the cylinder reversed.

A closer approximation to reality should be obtained by writing for u and v the corresponding values in the irrotational motion which most nearly coincides with the actual motion and regarding the difference as a small perturbation satisfying the equations (1.1). In this case also squares of small differences are neglected in the explicit expression for P . Of course, it is not at all obvious which irrotational

* Oseen himself did not put the equations of motion in the form (1.1), but Filon (*Proc. Roy. Soc. A*, 113 (1926), 7-27) has given the complete solution using this form. Filon, however, makes no attempt to satisfy boundary conditions at the cylinder.

motion is the one best suited in each particular case. This suggestion has been made by several investigators, including Boussinesq (1), Southwell and Squire (2), Zeilon (3), Burgers (4), Meksyn (5). But, as far as I am aware, the complete solution has been given in no case other than that which assumes the undisturbed uniform motion as the basic irrotational motion. This, of course, is the usual Oseen approximation. Southwell and Squire obtain five particular solutions of the equation satisfied by ω and satisfy the boundary conditions at five points only; Burgers neglects $\partial^2\omega/\partial\beta^2$ in the differential equation for ω ; while Zeilon contents himself with a particular solution and is concerned with the limiting case $\nu \rightarrow 0$ only.

Meksyn, however, though calculating the perturbation on the basis of Oseen's equations, superposes it on the irrotational motion appropriate to the problem of an infinite cylinder placed in a stream of non-viscous fluid, so that the boundary conditions are not quite the same as in Oseen's case. In his numerical illustration, however, he makes a grave error and his results cannot be relied on. This point will be referred to again in § 3.

The object of the present paper is to give the general solutions for circular and elliptic cylinders, taking as basic irrotational motion that which is appropriate to the problem of the cylinder placed in a uniform stream of non-viscous fluid. Considerable use is made of Meksyn's analytical methods, of which an essential feature is his use of Green's function for the calculation of the stream function. Here, however, Green's function is obtained by far more elementary methods.

2. The Modified Equations and their Transformation

Let us write

$$u = -\frac{\partial\alpha}{\partial x} = -\frac{\partial\beta}{\partial y}, \quad v = -\frac{\partial\alpha}{\partial y} = \frac{\partial\beta}{\partial x}. \quad (2.1)$$

Equations (1.1) now become

$$\left. \begin{aligned} -\frac{\partial\beta}{\partial x}\omega + \nu\frac{\partial\omega}{\partial y} &= -\frac{\partial P}{\partial x} \\ -\frac{\partial\beta}{\partial y}\omega - \nu\frac{\partial\omega}{\partial x} &= -\frac{\partial P}{\partial y} \end{aligned} \right\} \quad (2.2)$$

Introducing new variables (ξ, η) , which are not to be confused with

the elliptic coordinates (ξ, η) , introduced at the end of the present section, by means of the transformation

$$\zeta = f(z) \quad (2.3)$$

where $z = x + iy$, $\zeta = \xi + i\eta$,

it is easily seen that the above equations are equivalent to

$$\left. \begin{aligned} -\frac{\partial \beta}{\partial \xi} \omega + \nu \frac{\partial \omega}{\partial \eta} &= -\frac{\partial P}{\partial \xi} \\ -\frac{\partial \beta}{\partial \eta} \omega - \nu \frac{\partial \omega}{\partial \xi} &= -\frac{\partial P}{\partial \eta} \end{aligned} \right\} \quad (2.4)$$

i.e. the equations are invariant with respect to a transformation of the type (2.3). In particular, when

$$\xi + i\eta = \alpha + i\beta, \quad (2.5)$$

the equations take on a simplified form,

$$\left. \begin{aligned} \nu \frac{\partial \omega}{\partial \beta} &= -\frac{\partial P}{\partial \alpha} \\ -\omega - \nu \frac{\partial \omega}{\partial \alpha} &= -\frac{\partial P}{\partial \beta} \end{aligned} \right\} \quad (2.6)$$

which is precisely Oseen's form except that (α, β) replace (x, y) as independent variables.

It is easily verified that both ω and P satisfy the equation

$$\frac{\partial U}{\partial \alpha} + \nu \left(\frac{\partial^2 U}{\partial \alpha^2} + \frac{\partial^2 U}{\partial \beta^2} \right) = 0. \quad (2.7)$$

If one of them is known, the other can be found by quadrature.

For example

$$P = \int \left\{ -\nu \frac{\partial \omega}{\partial \beta} d\alpha + \left(\omega + \nu \frac{\partial \omega}{\partial \alpha} \right) d\beta \right\}. \quad (2.8)$$

Alternatively, both can be derived from a single function U , satisfying (2.7), by means of the relations

$$\omega = \frac{\partial U}{\partial \alpha}, \quad P = -\nu \frac{\partial U}{\partial \beta}. \quad (2.9)$$

The function U can be interpreted physically in the particular case

$$\alpha + i\beta = -V(x + iy),$$

i.e. in Oseen's approximation. In this case

$$\omega = -\frac{1}{V} \frac{\partial U}{\partial x} = -\frac{\nu}{V^2} \nabla^2 U = \nabla^2 \psi$$

where ψ is the stream function. It follows that $\nu U/V^2$ differs from ψ by a harmonic function.

Let us now write, in order to solve (2.8),

$$U = U' e^{-\alpha/2\nu}, \quad (2.10)$$

$$\text{so that } U' \text{ satisfies} \quad \frac{\partial^2 U'}{\partial \alpha^2} + \frac{\partial^2 U'}{\partial \beta^2} = \frac{U}{4\nu^2} \quad (2.11)$$

This equation can be solved in terms of Bessel functions of the second kind with imaginary argument. Unfortunately, these in general have singularities at points corresponding to the boundary of the cylinder and are therefore unsuitable. This is not the case in Oseen's approximation. By suitable choice of origin the singularity falls inside the cylinder and therefore causes no difficulty. It is true that the singularity can be removed by integration of certain solutions, appropriately weighted, over the boundary of the cylinder. This is the method of Bairstow, Cave and Lang (6), Burgers, Southwell and Squire, and others. Here, however, we avoid the singularity at the expense of obtaining the solution in terms of functions which have not been studied to the same extent as Bessel functions.

We make the transformation

$$\alpha + i\beta = -Vc \cosh \zeta. \quad (2.12)$$

We see at once that U' satisfies the equation

$$\frac{\partial^2 U'}{\partial \zeta^2} + \frac{\partial^2 U'}{\partial \eta^2} = k^2 (\cosh^2 \zeta - \cos^2 \eta) U', \quad (2.13)$$

where

$$k = Vc/2\nu.$$

This is a well-known equation and can be solved in terms of Mathieu functions and their associated functions. In this context (ξ, η) are elliptic coordinates in the (α, β) plane, not in the (x, y) plane.

3. General Expression for the Vorticity

$$\text{Writing} \quad U' = f(\xi)g(\eta) \quad (3.1)$$

and substituting in (2.13) we get the following equations for $f(\xi)$ and $g(\eta)$:

$$\left. \begin{aligned} \frac{d^2 f}{d\xi^2} - (k^2 \cosh^2 \xi + E)f &= 0 \\ \frac{d^2 g}{d\eta^2} + (k^2 \cos^2 \eta + E)g &= 0 \end{aligned} \right\}. \quad (3.2)$$

Solutions, periodic in η , are obtained for a discrete set of *eigen* values of E . They fall into four groups:*

$$\left. \begin{aligned} & \left. \begin{aligned} \text{ce}_{2n}(\eta) &= \sum_{r=0}^{\infty} A_{2r}^{2n} \cos 2r\eta, \\ \text{Cek}_{2n}(\xi) &= \sum_{r=0}^{\infty} A_{2r}^{2n} K_{2r}(k \cosh \xi) \end{aligned} \right\} \text{(I)} \\ & \left. \begin{aligned} \text{ce}_{2n+1}(\eta) &= \sum_{r=0}^{\infty} A_{2r+1}^{2n+1} \cos(2r+1)\eta, \\ \text{Cek}_{2n+1}(\xi) &= \sum_{r=0}^{\infty} A_{2r+1}^{2n+1} K_{2r+1}(k \cosh \xi) \end{aligned} \right\} \text{(II)} \\ & \left. \begin{aligned} \text{se}_{2n}(\eta) &= \sum_{r=0}^{\infty} B_{2r}^{2n} \sin 2r\eta, \\ \text{Sek}_{2n}(\xi) &= \tanh \xi \sum_{r=0}^{\infty} 2r B_{2r}^{2n} K_{2r}(k \cosh \xi) \end{aligned} \right\} \text{(III)} \\ & \left. \begin{aligned} \text{se}_{2n+1}(\eta) &= \sum_{r=0}^{\infty} B_{2r+1}^{2n+1} \sin(2r+1)\eta, \\ \text{Sek}_{2n+1}(\xi) &= \tanh \xi \sum_{r=0}^{\infty} (2r+1) B_{2r+1}^{2n+1} K_{2r+1}(k \cosh \xi) \end{aligned} \right\} \text{(IV)} \end{aligned} \right\} \quad (3.3)$$

where $K_n(x)$ is Bessel's function, with imaginary argument, of the second kind. Its asymptotic expansion is

$$K_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)} e^{-x} \left(1 - \frac{1}{8x} + \frac{n^2}{2} \frac{1}{x} + \dots\right). \quad (3.4)$$

The A_r^n and B_r^n are determinate constants.

For brevity we shall write

$$\left. \begin{aligned} C_n &= e^{k \cosh \xi \cos \eta} \text{Cek}_n(\xi) \text{ce}_n(\eta) \\ S_n &= e^{k \cosh \xi \cos \eta} \text{Sek}_n(\xi) \text{se}_n(\eta) \end{aligned} \right\}. \quad (3.5)$$

Each of the functions C_n and S_n satisfies the differential equation satisfied by ω and P . But, as shown in the appendix, each of the C_n functions gives rise to infinite circulation at infinity and hence to infinite lifting force on the cylinder. This suggests that linear combinations of C_n which do not give rise to infinite circulation should be taken as elementary solutions instead of the C_n themselves.

We propose to normalize the $\text{ce}_n(\eta)$ functions so that

$$\sum_{r=0}^{\infty} A_{2r}^{2n} = 1, \quad \sum_{r=0}^{\infty} A_{2r+1}^{2n+1} = 1. \quad (3.6)$$

(The usual procedure is to make A_{2n}^{2n} , A_{2n+1}^{2n+1} equal to unity.)

* Meksyn, loc. cit. (3.26).

If now we define functions

$$C_{n,n+1} = C_n - C_{n+1}, \quad (3.7)$$

it will be shown in the appendix that these do not give rise to infinite circulation, and can therefore be taken as elementary solutions for ω . The S_n functions are odd functions of η and give rise to zero circulation, and so no special precaution need be taken at this stage. The general expression for ω is therefore

$$\omega = \sum_{n=0}^{\infty} \gamma_n C_{n,n+1} + \sum_{n=1}^{\infty} \sigma_n C_n, \quad (3.8)$$

where γ_n and σ_n are arbitrary constants.

(The numerical example worked out by Meksyn involves but one C_n function, and hence infinite circulation. Meksyn failed to appreciate this fact owing to an error in replacing the exponential factor by unity in his numerical illustration.* This factor† is

$$\begin{aligned} \exp\left\{\frac{Vc}{2\nu}(\cosh \xi \cos \eta \cos \theta + \sinh \xi \sin \eta \sin \theta)\right\} \\ = \exp\left\{\frac{Vc}{4\nu}e^{\xi} \cos(\eta - \theta)\right\} \quad \text{for large } \xi. \end{aligned}$$

It becomes infinite with ξ for all values of $|\eta - \theta| < \frac{1}{2}\pi$, and there is no justification for replacing it by unity even for very small values of the Reynolds number. Meksyn calculates the case for which $Vc/\nu = 1$.)

In principle, the corresponding expression for P can be obtained by quadrature. Using the relations (2.4) we get

$$P = \int \left\{ -e^{2k \cosh \xi \cos \eta} \left[\frac{\partial}{\partial \eta} (\omega e^{-2k \cosh \xi \cos \eta}) d\xi - \frac{\partial}{\partial \xi} (\omega e^{-2k \cosh \xi \cos \eta}) d\eta \right] \right\}. \quad (3.9)$$

The integration, though laborious, can be performed.

The arbitrary constants γ_n and σ_n are determined from the boundary conditions at the cylinder. But first of all it will be convenient to determine the stream function.

4. The Stream Function

Let the stream function be

$$\psi = \psi_1 + \beta, \quad (4.1)$$

where ψ_1 is due to the vorticity. The normal and tangential

* Loc. cit. (5.12).

† Loc. cit. (3.28).

components of the velocity at the cylinder are

$$-h \frac{\partial \psi}{\partial \eta}, \quad h \frac{\partial \psi}{\partial \xi}, \quad \text{with } \xi = 0, \quad (4.2)$$

where

$$h^{-1} = \left| \frac{dz}{d\zeta} \right|. \quad (4.3)$$

These components have to vanish. Now ψ_1 satisfies the differential equation

$$\frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} = h^{-2} \omega. \quad (4.4)$$

Following Meksyn, we solve this equation with the aid of Green's function for the semi-infinite strip bounded by

$$\eta = 0, \quad \eta = 2\pi, \quad \xi = 0. \quad (4.5)$$

This function must be periodic in η (with period 2π), and vanish for $\xi = 0$. Meksyn obtains a series for it. It is much simpler to transform the unit circle into the given boundary—the transform of Green's function for the unit circle is the function required.

Write $z' = e^\zeta$. (4.6)

This transformation transforms the space external to the circle $|z'| = 1$ into the region defined by (4.5). Green's function, having a singularity at the point z'_1 external to the unit circle, is

$$G = R \left\{ \frac{1}{2\pi} \log(z' - z'_1) / z'_1(z' - z'_2) \right\}, \quad (4.7)$$

where $\bar{z}'_1 z'_2 = 1, \quad z'_1 \bar{z}'_1 > 1.$

Writing $z'_1 = e^{\zeta_1}, \quad z'_2 = e^{-\bar{\zeta}_1}$ (4.8)

we finally get $G = R \left\{ \frac{1}{2\pi} \log(e^\zeta - e^{\zeta_1}) / (e^{\zeta + \bar{\zeta}_1} - 1) \right\}.$ (4.9)

This function is symmetrical in (ξ, η) and (ξ_1, η_1) . By a well-known theorem

$$\psi_1 = \int_0^\infty \int_0^{2\pi} \omega(\xi_1, \eta_1) G(\xi, \eta; \xi_1, \eta_1) h_1^{-2} d\xi_1 d\eta_1. \quad (4.10)$$

G , and hence ψ_1 and $\partial \psi_1 / \partial \eta$, are zero for $\xi = 0$, i.e. the normal velocity is zero at the cylinder as required. This condition is not violated if

we add to ψ_1 the term $\frac{K}{2\pi} \xi$, so that the total stream function can be taken as

$$\psi = \psi_1 + \beta + \frac{K}{2\pi} \xi. \quad (4.11)$$

The tangential component of the velocity will vanish provided that

$$\frac{\partial \psi_1}{\partial \xi} = -\frac{\partial \beta}{\partial \xi} - \frac{K}{2\pi} \quad (4.12)$$

when $\xi = 0$, i.e. provided that

$$\int_0^\infty \int_0^{2\pi} \omega(\xi_1, \eta_1) \frac{\partial G}{\partial \xi} \Big|_{\xi=0} h_1^{-2} d\xi_1 d\eta_1 = Vc \sin \eta - \frac{K}{2\pi}. \quad (4.13)$$

Expanding the right-hand side of (4.7) in the neighbourhood of $|z'| = 1$ we get

$$\begin{aligned} G &= -R \left[\frac{1}{2\pi} \left\{ \sum_1^\infty \frac{1}{n} \left(\frac{z_1'^n}{z_1'^n} - \frac{z_2'^n}{z_2'^n} \right) + \log z' \right\} \right] \\ &= -\frac{1}{2\pi} \sum_1^\infty \frac{1}{n} \{ (e^{n(\xi-\xi_1)} - e^{-n(\xi+\xi_1)}) \cos n(\eta-\eta_1) \} - \frac{\xi}{2\pi}. \end{aligned} \quad (4.14)$$

It follows that

$$\frac{\partial G'}{\partial \xi} \Big|_{\xi=0} = -\frac{1}{2\pi} \left\{ 1 + 2 \sum_1^\infty e^{-n\xi_1} \cos n(\eta-\eta_1) \right\} \quad (4.15)$$

which is identical with Meksyn's form (4.17) when his $\xi_0 = 0$.

Substituting this expression in (4.13) and equating coefficients of $\cos n\eta$, $\sin n\eta$ on both sides of the equation we get the following relations:

$$\left. \begin{aligned} \int_0^\infty \int_0^{2\pi} \omega(\xi, \eta) h^{-2} d\xi d\eta &= K \\ \int_0^\infty \int_0^{2\pi} \omega(\xi, \eta) e^{-n\xi} \cos n\eta h^{-2} d\xi d\eta &= 0 \quad (n > 0) \\ \int_0^\infty \int_0^{2\pi} \omega(\xi, \eta) e^{-\xi} \sin \eta h^{-2} d\xi d\eta &= Vc \\ \int_0^\infty \int_0^{2\pi} \omega(\xi, \eta) e^{-n\xi} \sin n\eta h^{-2} d\xi d\eta &= 0 \quad (n > 1) \end{aligned} \right\}. \quad (4.16)$$

These constitute a doubly infinite set of linear equations in the constants γ_n , σ_n .

5. The Circular Cylinder

If a cylinder of radius unity is put in a uniform stream of non-viscous liquid, the resulting motion is given by

$$\alpha + i\beta = -V \left(z + \frac{1}{z} \right) = -2V \cosh \zeta, \quad (5.1)$$

the adjustable constant c of (2.12) being here 2. This gives

$$z = e^{\zeta} \quad (5.2)$$

and

$$h^{-2} = \left| \frac{dz}{d\zeta} \right|^2 = e^{2\zeta}. \quad (5.3)$$

On account of the symmetry we must have in this case

$$\kappa = 0, \quad \gamma_n = 0. \quad (5.4)$$

The σ_m are determined from the equations

$$\sum_{m=1}^{\infty} \sigma_m \int_0^{\infty} \int_0^{2\pi} S_m e^{\xi} \sin \eta \, d\xi d\eta = 2V,$$

$$\sum_{m=1}^{\infty} \sigma_m \int_0^{\infty} \int_0^{2\pi} S_m e^{-(n-2)\xi} \sin n\eta \, d\xi d\eta = 0 \quad (n > 1). \quad (5.5)$$

6. The Elliptic Cylinder and the Flat Plate

It is easily verified that the equations

$$\alpha + i\beta = -Vc \cosh \zeta, \quad (6.1)$$

$$z = c \cosh(\zeta + \zeta_0) \quad (6.2)$$

represent the irrotational motion due to an elliptic cylinder placed in a stream of non-viscous liquid, the major axis being inclined at an angle η_0 to the general direction of the stream. The equation of the cylinder is

$$\xi = 0, \quad (6.3)$$

which is coincident with part of the streamline $\beta = 0$. This is obvious on resolving (6.1) into its real and imaginary parts. In this case

$$h^{-2} = c^2 \{ \cosh^2(\xi + \xi_0) - \cos^2(\eta + \eta_0) \}. \quad (6.4)$$

The circular cylinder can be considered as a particular case, as can also the flat plate, which is the limiting case

$$\xi_0 = 0. \quad (6.5)$$

For a flat plate placed parallel to the stream, we have in addition

$$\eta_0 = 0,$$

and hence

$$h^{-2} = c^2 (\cosh^2 \xi - \cos^2 \eta). \quad (6.6)$$

In this case and in all cases of symmetry

$$\kappa = 0, \quad \gamma_n = 0,$$

which leaves a single infinity of linear equations in the constants σ_n .

The solutions for the general cylinder, numerical illustrations and formulae for lift and drag are reserved for another occasion.*

Appendix

Let ξ_1 be very large so that the equation

$$\xi = \xi_1 \quad (1)$$

represents a large contour remote from the origin. It can be shown that the function C_n gives rise to an infinite difference between the circulation over this contour and the contour at infinity,

$$\text{i.e.} \quad \kappa \Big|_{\xi_1}^{\infty} = \int_{\xi_1}^{\infty} \int_0^{2\pi} C_n h^{-2} d\xi d\eta \quad (2)$$

is infinite.

It will be assumed that ξ_1 is sufficiently large to justify replacing the K_n functions by their asymptotic expansions. Again, it will be sufficient to write

$$h^{-2} = e^{2\xi} = r^2,$$

$$\cosh \xi = \frac{1}{2}r.$$

The asymptotic expansion for K_n now becomes

$$K_n(\frac{1}{2}kr) = \sqrt{\left(\frac{\pi}{kr}\right)} e^{-kr/2} \left\{ 1 - \frac{1}{4kr} + \frac{n^2}{kr} + \dots \right\}. \quad (3)$$

For definiteness let us consider even functions C_{2n} . Then

$$\begin{aligned} \kappa \Big|_{\xi_1}^{\infty} = & \int_{\xi_1}^{\infty} \int_0^{2\pi} \sum_{s=0}^{\infty} A_{2s}^{2n} \sqrt{\left(\frac{\pi}{kr}\right)} e^{-kr \sin^2(\eta/2)} \left\{ 1 - \frac{1}{4kr} + \frac{4s^2}{kr} + \dots \right\} \times \\ & \times \sum_{s=0}^{\infty} A_{2s}^{2n} \cos 2s\eta r \, dr d\eta. \quad (4) \end{aligned}$$

For large values of r the exponential term obliterates the integrand except in an interval of order $1/\sqrt{(kr)}$ in the neighbourhood of $\eta = 0$. Adopting the usual procedure we introduce a new variable

$$\theta = \frac{1}{2}\sqrt{(kr)}\eta, \quad (5)$$

whose limits are $\pm \infty$ (since r is always large).

$$\begin{aligned} \kappa \Big|_{\xi_1}^{\infty} = & \int_{\xi_1}^{\infty} \int_{-\infty}^{\infty} \sum_{s=0}^{\infty} A_{2s}^{2n} \sqrt{\left(\frac{\pi}{k}\right)} e^{-\theta^2} \left\{ 1 - \frac{1}{4kr} + \frac{4s^2}{kr} + \dots \right\} \times \\ & \times \sum_{s=0}^{\infty} A_{2s}^{2n} \cos \frac{4s\theta}{\sqrt{(kr)}} \frac{2}{\sqrt{k}} \, dr d\theta. \quad (6) \end{aligned}$$

* I have to acknowledge the helpful criticisms and suggestions of a referee.

If we neglect terms of order r^{-2} in $\cos \frac{4s\theta}{\sqrt{(kr)}}$ we get

$$\kappa \Big|_{\xi_1}^{\infty} = \frac{2}{k} \sqrt{\pi} \int_{r_1}^{\infty} \int_{-\infty}^{\infty} \sum_{s=0}^{\infty} A_{2s}^{2n} e^{-\theta^2} \left(1 - \frac{1}{4kr} + \frac{4s^2}{kr} + \dots \right) \times \\ \times \sum_{s=0}^{\infty} A_{2s}^{2n} \left(1 - \frac{8s^2\theta^2}{kr} + \dots \right) dr d\theta. \quad (7)$$

Bearing in mind that

$$\int_{-\infty}^{\infty} \theta^2 e^{-\theta^2} d\theta = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\theta^2} d\theta,$$

and normalizing the Mathieu functions by means of the relation

$$\sum_{s=0}^{\infty} A_{2s}^{2n} = 1,$$

the difference in circulation becomes

$$\kappa \Big|_{\xi_1}^{\infty} = \frac{2\sqrt{\pi}}{k} \int_{r_1}^{\infty} \int_{-\infty}^{\infty} e^{-\theta^2} \left(1 - \frac{1}{4kr} + \dots \right) dr d\theta = \frac{2\pi}{k} \int_{\xi_1}^{\infty} \left(1 - \frac{1}{4kr} + \dots \right) dr. \quad (8)$$

This is infinite but independent of n . It is, however, clear that

$$C_{n,n+1} = C_n - C_{n+1},$$

would give rise to terms of order r^{-2} at most in the integrand of (8), and hence the difficulty of infinite circulation is avoided.

The above demonstration is obviously not rigorous, for it takes no account of the fact that s assumes all integral values up to infinity. A rigorous proof is desirable and no doubt possible; but very large values of s need not concern us as practical applied mathematicians, for, if the solutions are to be of value in practice, it must be possible to stop short at a finite value of s without appreciable error in the result.

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By CHAO KO (*Manchester*)

It was proved by Mordell in 1932 that every positive definite quadratic form in n variables with rational coefficients could be expressed as a sum of $n+3$ squares of linear forms with rational coefficients. He also proved that this was the best possible value when $n = 2$.*

1. It is sufficient to show that there exists a special form in n variables, which cannot be expressed as a sum of $n+2$ squares of linear forms with rational coefficients.

where b is any number which is not the sum of three rational squares, e.g. $4^\alpha(8\beta+7)$, where α and β are positive integers or zero. Suppose f can be written as

where the α 's are rational numbers. Then

Let $\rho_i = \pm 1$ ($i = 1, \dots, n-1$) be at our disposal. Consider the system of linear equations

$$\left. \begin{aligned} (a_{11} + \rho_1)x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0, \\ a_{21}x_1 + (a_{22} + \rho_2)x_2 + \dots + a_{2n}x_n &= 0, \\ * & * * * * * \\ a_{n-1,1}x_1 + a_{n-1,2}x_2 + \dots + (a_{n-1,n-1} + \rho_{n-1})x_{n-1} + a_{n-1,n}x_n &= 0. \end{aligned} \right\} (2)$$

† Ko, *Quart. J. of Math.* (Oxford), 8 (1937), 81-98.

Write

$$D(\rho_1, \dots, \rho_i) = \begin{vmatrix} a_{11} + \rho_1 & a_{12} & \cdot & \cdot & a_{1i} \\ a_{21} & a_{22} + \rho_2 & \cdot & \cdot & a_{2i} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i1} & a_{i2} & \cdot & \cdot & a_{ii} + \rho_i \end{vmatrix} \quad (i = 1, \dots, n-1).$$

Suppose now that one of the determinants $D(\rho_1, \dots, \rho_{n-1})$ is not zero. Then for this particular set $\rho_1, \dots, \rho_{n-1}$, there exists a set of solutions of (2) with $x_n = D(\rho_1, \dots, \rho_{n-1}) \neq 0$. Hence, from (1), b can be expressed as a sum of three rational squares, which is a contradiction.

We now show that the determinants $D(\rho_1, \dots, \rho_{n-1})$ cannot all vanish. For, if $D(\rho_1, \dots, \rho_{n-1}) = 0$ for all sets of the ρ 's, we have for any set $\rho_1, \dots, \rho_{n-2}$,

$$D(\rho_1, \dots, \rho_{n-2}, 1) = D(\rho_1, \dots, \rho_{n-2}, -1) = 0,$$

so that, on subtraction,

$$D(\rho_1, \dots, \rho_{n-2}) = 0$$

for every set of the ρ 's. Proceeding in this way, we shall obtain ultimately

$$D(\rho_1) = a_{11} + \rho_1 = 0,$$

i.e.

$$a_{11} + 1 = 0, \quad a_{11} - 1 = 0,$$

which is a contradiction and our theorem is proved.

2. When $n = 2$, the condition for a form to be representable as a sum of $n+2$ squares depends only on the value of its determinant. It is interesting to note that this is no longer true when $n > 2$, e.g. the form

$$g = 3(x_1^2 + x_2^2 + x_3^2) + 4(x_1x_2 + x_2x_3 + x_3x_1)$$

with determinant $\begin{vmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{vmatrix} = 7$ can be expressed as a sum of five squares

$$g = x_1^2 + x_2^2 + x_3^2 + 2(x_1 + x_2 + x_3)^2,$$

and hence in general the form

$$g + \sum_{i=4}^n x_i^2$$

with determinant 7 can be expressed as a sum of $n+2$ squares of linear forms with integer coefficients.

In closing I should like to thank Professor Mordell for his helpful criticism.

ON CERTAIN CHAINS OF GEOMETRICAL THEOREMS

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1. It is known that the intercepts made upon any line by a hyperbola and by its asymptotes have the same middle point. It follows that, if one straight line and two points are given in a plane, and hyperbolas are drawn having the line as an asymptote and passing through the points, the second asymptotes of all the hyperbolas will pass through a certain point, which will be called the point $P(1, 2)$. It lies on the line joining the given points and is easily determined when the line and points are given. If one straight line and three points are given, each two points determine a point $P(1, 2)$, and these three points must lie on a line which will be called the line $L(1, 3)$. $L(1, 3)$ is plainly the second asymptote of the hyperbola which has the given line as one asymptote and passes through the given points. From this follows an unending chain of theorems concerning points and lines.

- (a) 1 line and 2 points define as above a point called the point $P(1, 2)$.
- (b) 1 line and 3 points define as above a line called the line $L(1, 3)$.
- (c) 2 lines and 3 points define 2 such lines as $L(1, 3)$ in (b). These meet in a point. Call this point $P(2, 3)$.
- (d) 2 lines and 4 points define 4 such points as $P(2, 3)$ in (c). These lie upon a line. Call this line $L(2, 4)$.
- (e) 3 lines and 4 points define 3 such lines as $L(2, 4)$ in (d). These meet in a point. Call this point $P(3, 4)$.
- (f) 3 lines and 5 points define 5 such points as $P(3, 4)$ in (e). These lie upon a line. Call this line $L(3, 5)$.
- (g) 4 lines and 5 points define 4 such lines as $L(3, 5)$ in (f). These meet in a point. Call this point $P(4, 5)$.

.

The sequence may be continued indefinitely. At each stage the number of points or lines (alternately) is increased by one—from $r-1$ to r , let us say. By omitting each of the r in turn, r examples of the theorem of the preceding stage are obtained and r points or

lines are defined. If lines, it will be shown that they are concurrent; if points, collinear: and so the chain is carried on.

2. Proof. Each step in the following proof could be explained in geometrical language and coordinates might be dispensed with; but they are convenient for comments and references. The cartesian x, y will serve, but it will be advantageous to make the formulae homogeneous by introducing z (z being 1). The given lines have equations $L_1 = 0, L_2 = 0, L_3 = 0, \dots$; an unknown line will have an equation $\lambda = 0$, L and λ being linear functions of x, y, z , viz.

$$L_1 \equiv a_1x + b_1y + c_1z, \text{ etc.}; \quad \lambda \equiv \alpha x + \beta y + \gamma z. \quad (\text{i})$$

For stages (a) and (b) consider the family of conics

$$K_1/L_1 + \lambda/(z^2) = 0. \quad (\text{ii})$$

Whatever the values of $\alpha, \beta, \gamma, K_1$, the conic has the given line L_1 for one asymptote and the unknown line λ for the other. If, as in (b), three points of the conic are given, the conic and the line λ are definite; λ is the line $L(1, 3)$. But if, as in (a), only two points are given, one degree of freedom is left and the line λ must pass through a certain fixed point, the point $P(1, 2)$.

For stages (c) and (d) in which two lines L_1 and L_2 appear, consider the family of curves of order three:

$$K_1/L_1 + K_2/L_2 + \lambda/(z^2) = 0. \quad (\text{iii})$$

Whatever the values of $\alpha, \beta, \gamma, K_1, K_2$, the curve has the given lines L_1, L_2 as asymptotes and passes through their intersection; it has a third asymptote λ . Four conditions are required to fix the curve. If the curve is made to pass through the three points used in (b), it will have one degree of freedom left and λ will have to pass through a certain fixed point. Now the family of cubics includes one for which K_2 vanishes; this curve breaks up into the line L_2 and a conic which can only be the conic of stage (b) that appeared in the last paragraph. The fixed point therefore lies on the line $L(1, 3)$ of stage (b), and on the line derived in the same manner from L_2 and the three points: it is the point that was called $P(2, 3)$ in (c). Again, if the cubic is made to pass through one further point, four points in all, it and the line λ will be fixed. Moreover, λ must be the line called $L(2, 4)$ in (d), because it passes through the point $P(2, 3)$ just found, and three more such points derived from L_1, L_2 , and other sets of three of the four points.

In stages (e) and (f) three lines, L_1 , L_2 , L_3 , and either four or five points enter. Proofs are obtained from the quartic curves

$$K_1/L_1 + K_2/L_2 + K_3/L_3 + \lambda/(z^2) = 0 \quad (\text{iv})$$

which pass through the intersections of each two of L_1 , L_2 , L_3 , and have these lines as asymptotes; each has a fourth asymptote λ . If the curve is made to pass through the four points used in (d), the line λ will pass through a fixed point. But one curve of the pencil has $K_3 = 0$ and breaks up into the line L_3 and the cubic used in the proof of (d). Hence the fixed point lies on the line $L(2, 4)$ discovered in (d), and on the two similar lines derived from these four points and another pair of lines selected from L_1 , L_2 , L_3 . The three lines $L(2, 4)$ are therefore concurrent, and the fixed point is the point $P(3, 4)$ of (e). Again, if the quartic is made to pass through five points, the four just used and another, the curve and the line λ are definitely fixed. But the quartic belongs to the pencil just discussed and the point $P(3, 4)$ must lie on λ ; as must the points $P(3, 4)$ derived from these three lines L_1 , L_2 , L_3 and any four of the five points: λ here is the line $L(3, 5)$ of (f).

In this way the lines and points are successively identified with those named in § 1 and their properties of collinearity and concurrence are proved.

3. Consideration of these results. It will be noticed that in the drawing of this elaborate figure only one construction is wanted, that of determining the point $P(1, 2)$ when a line and two points are given: this construction would have to be used repeatedly. Apart from this all the lines are determined by the simple process of joining known points, for which nothing but a straight edge is required; and these lines by their intersections define further points.

3.1. It is in the first stage (a) that extensions of the results should be looked for. In whatever coordinates are used, the last term of equations (ii), (iii), (iv) of § 2 has for its denominator the square of the line at infinity. It might equally well be any line without affecting the argument; the word asymptote would have to be dropped. But the results could be obtained by projection.

Nothing in the reasoning of § 2 turns upon the fact that the last denominator in these equations is the square of a linear form; it may equally well be a product of two linear forms or a wholly unrestricted quadratic form $Q(x, y, z)$. A genuine extension springs from this fact,

viz. that a chain of theorems exactly as enunciated in § 1 may be derived from any conic Q in the plane. The only desideratum is a construction for the point $P(1, 2)$ in connexion with the fundamental conic Q , when (z^2) is replaced by $Q(x, y, z)$ in (ii). A line L_1 and a pair of points are given, and on the line joining the latter a second pair of points is cut out by Q . The two pairs of points define an involution on the line, and in this involution $P(1, 2)$ is the partner of the point where L_1 meets the line. The statement of the theorems in § 1 needs no modification; but the curves in § 2 now pass through the intersections of the lines L_1, L_2, L_3, \dots with Q and with one another: the line λ joins the two further points in which the curve meets Q .

3.2. With the introduction of Q the theorems become projective;* but its presence detracts from the interest of the geometry. If Q could be taken as the Absolute Conic the theorems would assume a simple metrical form. The metrical geometry of the surface of a sphere depends upon such a conic. Interpreting x, y, z in § 2 as cartesian coordinates of a point on a sphere with centre at the origin, $Q(x, y, z)$ is $x^2 + y^2 + z^2$, and the construction for the point $P(1, 2)$ is as nearly identical with that of § 1 as it can be. If the two given points are E, F , and if L_1 meets the line (great circle) EF in H , the segments $E-F$ and $H-P(1, 2)$ have the same middle point. Therefore:

The chain of theorems enunciated in § 1 holds on a spherical surface as well as on a plane.

4. 4.1. **Duality.** This principle shows that similar chains of theorems, with the roles of points and lines interchanged, may be developed from a first stage of one given point and two given lines. To adapt the equations of 3.1 (in which the denominator z^2 of § 2 has given place to Q), the coordinates (l, m, n) of lines may be substituted for (x, y, z) . The analogues of the projective properties when K is unrestricted, and those of the metrical properties on a spherical surface, need not be stated; they are obtained by the standard methods. But in the euclidean plane a chain of metrical theorems, which cannot be derived (by duality or by any other method) from those proved in § 2, is brought to light.

* It is not correct to describe the theorems in § 1 merely as metrical. They are not projective, but belong to what Euler termed 'Affine' geometry, in which the properties persist after *orthogonal* projection. *Encyklopädie der Math. Wiss.*, Bd. 3, Teil 1, 909.

4.2. With (l, m, n) in place of (x, y, z) the linear forms L_1, L_2, \dots, λ equated to 0 define given points S_1, S_2, \dots and an unknown point Σ . If it is assumed that rectangular cartesian axes are used, the Absolute Conic, which breaks into linear factors representing the circular points I, J , is $l^2 + m^2 = 0$. But, as in § 2, coordinates are convenient, not essential.

Thus modified, equation (ii) of § 2 is replaced by

$$K_1/L_1 + \lambda/(l^2 + m^2) = 0.$$

This is the tangential equation of a conic having the given point S_1 and the unknown point Σ as foci. If the conic is made to touch two given lines, the locus of Σ is a line through their point of intersection. If the conic is made to touch three given lines, Σ is fixed: it is called the *isogonal conjugate** of S_1 in the triangle formed by the lines. At the next stage of the proof curves of class 3 having S_1, S_2 as foci and touching the line $S_1 S_2$ are used; these are made to touch either three or four other given lines. . . . But curves of higher class are not needed for the construction of the points and lines of the chain. Indeed, since the isogonal conjugate point is well known, it makes the enunciation simpler to start from one point and three lines, suppressing the first stage. Thus the chain of theorems is:

- 1 point and 3 lines define an isogonal conjugate point, here called $P(1, 3)$.
- 2 points and 3 lines define 2 isogonal conjugate points. Join these points. Call this line $L(2, 3)$.
- 2 points and 4 lines define 4 such lines as $L(2, 3)$. These must meet in a point. Call this point $P(2, 4)$.
- 3 points and 4 lines define 3 such points as $P(2, 4)$. These must lie upon a line. Call this line $L(3, 4)$.
- 3 points and 5 lines define 5 such lines as $L(3, 4)$. These must meet in a point. Call this point $P(3, 5)$.

.

and so on indefinitely.

To sum up: *Two chains of theorems, one affine, the other metrical, exist in ordinary plane geometry; and each chain is applicable, without verbal change, to the metrical geometry of a spherical surface.* [It might

* See for example Durell, *Plane Geometry for Advanced Students*, Part 1, p. 50.

be added that the chains are valid in non-euclidean plane geometry, without reservation in the Riemann form; in the hyperbolic form the difficulty arises that two lines may or may not intersect within the plane. A projective form of the theorems requires reference to a fundamental conic.]

5. Further extensions. The chains of theorems and the methods of proof may be extended to geometries of three or more dimensions. It appears unnecessary to pursue this, but the extension of the second chain in § 4 to euclidean geometry of three dimensions is sketched below.

Given a tetrahedron and a point, a quadric of revolution can be found having the point as focus and touching the faces of the tetrahedron; the second focus of the quadric will be called the point $P(1, 4)$. $P(1, 4)$ might be described as the isogonal conjugate of the given point, or as the centre of the sphere which passes through the reflections of the given point in the faces of the tetrahedron. To construct the point $P(1, 4)$ when a point and four planes are given is the only construction needed in what follows.

- 1 point and 4 planes define a point $P(1, 4)$ as has been explained.
- 2 points and 4 planes define 2 such points as $P(1, 4)$. Call the line which joins them $L(2, 4)$.
- 2 points and 5 planes define 5 such lines as $L(2, 4)$. These must meet in a point. Call this point $P(2, 5)$.
- 3 points and 4 planes define 3 such lines as $L(2, 4)$. These must lie in a plane. Call this plane $F(3, 4)$.
- 3 points and 5 planes define 3 such points as $P(2, 5)$ and 5 such planes as $F(3, 4)$. These lie or meet in a line. Call the line $L(3, 5)$.
- 3 points and 6 planes define 6 such lines as $L(3, 5)$. These must meet in a point. Call this point $P(3, 6)$.
- 4 points and 5 planes define 4 such lines as $L(3, 5)$. These must lie in a plane. Call this plane $F(4, 5)$.
- 4 points and 6 planes define 4 such points as $P(3, 6)$ and 6 such planes as $F(4, 5)$. These lie or meet in a line. Call this line $L(4, 6)$.
- 4 points and 7 planes define 7 such lines as $L(4, 6)$. These must meet in a point. Call this point $P(4, 7)$.

The chain of theorems can be continued indefinitely: it is only

necessary to increase every number by 1 in order to reach the next stage. Stages in which fewer than four planes enter have been dropped. Not only do 7 lines $L(4, 6)$ pass through the point $P(4, 7)$ last defined, but 21 planes $F(4, 5)$; however, since each plane contains two of the seven concurrent lines, they hardly deserve mention.

The proofs follow the course of those used in § 4. In the stages when four points are given, the last three stages discussed above, we consider surfaces of class 5 included in

$$K_1/L_1 + K_2/L_2 + K_3/L_3 + K_4/L_4 + \lambda/(l^2 + m^2 + n^2) = 0,$$

$L_1 = 0$, $L_2 = 0$, $L_3 = 0$, $L_4 = 0$, and $\lambda = 0$, are the equations of the four given and of one unknown point; $l^2 + m^2 + n^2 = 0$ is the Absolute, if rectangular cartesian coordinates are used. Seven given tangent planes define such a surface and fix the point λ ; six leave λ free to move on a line; five leave it free to move in a plane.

UNBESCHRÄNKTE OPERATOREN, DIE EINE VERALLGEMEINERUNG DER FOURIER- TRANSFORMATIONEN SIND

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1. DIE hier auftretenden Transformationen sollen in \mathfrak{L}^2 , dem Raume der quadratisch von Null bis Unendlich im Lebesgueschen Sinne integrierbaren Funktionen, sein.

Bekannt ist die Plancherelsche† Theorie der Transformationen

$$g(x) = \frac{d}{dx} \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^{\infty} \frac{\psi_t(tx)}{t} f(t) dt, \quad \psi_1(u) = \sin u = \int_0^u \cos v dv,$$

$$\psi_2(u) = 1 - \cos u = \int_0^u \sin v dv, \quad (1)$$

und die Watsonsche‡ Verallgemeinerung

$$g(x) = \frac{d}{dx} \int_0^{\infty} \frac{\chi(tx)}{t} f(t) dt, \quad f(x) = \frac{d}{dx} \int_0^{\infty} \frac{\chi(tx)}{t} g(t) dt. \quad (1.1)$$

Die dort betrachteten Transformationen sind beschränkt. Wählt man

$$\chi_{\alpha}(u) = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \int_0^u \left(\frac{v}{2} \right)^{-\alpha} \cos v dv, \quad g(x) = \frac{d}{dx} \int_0^{\infty} \frac{\chi_{\alpha}(tx)}{t} f(t) dt, \quad (1.2)$$

so erhält man für $0 \leq \alpha < \frac{1}{2}$ wieder eine beschränkte Transformation, die für alle $f(t) \in \mathfrak{L}^2$ Sinn hat. Die Frage nach der Umkehrung nun führt auf eine Transformation derselben Gestalt wie (1.2), nur tritt $-\alpha$ an die Stelle von α . Und diese Transformation hat keineswegs mehr für alle $f(t) \in \mathfrak{L}^2$ Sinn; sie ist auch nicht mehr beschränkt. Die Untersuchung dieser und einer allgemeineren§ Operation $T[\chi]$, die ein einfaches, vollständig übersichtliches Beispiel eines unbeschränkten Operators darstellt, ist die Aufgabe der vorliegenden Arbeit, gleichzeitig als notwendige Grundlage für die Lösung weiterer

† *Rend. di Palermo*, 30 (1910), 289–335, fortan als Plancherel I bezeichnet. Das Gleichheitszeichen soll hier, wie in allen ähnlichen Fällen, immer Gleichheit bis auf eine Menge vom Masze 0 bedeuten.

‡ *Proc. London Math. Soc.* (2) 35, (1933), 156–99.

§ Cf. Verfasser, *Quart. J. of Math.* (Oxford) 31 (1937), 172–85. Die Sätze aus dieser Arbeit werden hier durch den Zusatz V. kenntlich gemacht. Vergleiche z. B. V. Satz 3 und hier Satz K, der schärfer ist.

Fragen: besonders der recht lohnenden Frage nach Auftreten und Eigenschaften von Eigenwerten und selbstreziproken Funktionen.

Im nächsten Abschnitt dieser Arbeit werden wesentliche Eigenschaften der Operatoren $T[\chi]$ untersucht, im dritten wird besonders die Frage der Umkehrbarkeit erörtert, im vierten werden gewisse Operatoren behandelt, die als Produkt zweier der vorigen Art darstellbar sind, und im letzten Abschnitt Beispiele gegeben.

Als Definitionsbereich oder Bereich ('domain') \mathfrak{D} einer Transformation T bezeichnet man die Gesamtheit aller $f(x) \in \mathfrak{L}^2$, denen durch T je eine Funktion $g(x) \in \mathfrak{L}^2$ zugeordnet wird, als Wertevorrat oder Vorrat ('range') \mathfrak{R} die Gesamtheit aller dieser zugeordneten $g(x)$. T heisst umkehrbar, wenn diese Zuordnung ein-eindeutig, T heisst schlechthin umkehrbar, wenn ausserdem der Vorrat \mathfrak{R} gleich \mathfrak{L}^2 ist. Es bedeute ferner $f_n(x) \rightarrow f(x)$ Konvergenz der Folge f_1, f_2, \dots gegen f im Quadratmittel, also

$$\|f - f_n\| = \left(\int_0^\infty |f(x) - f_n(x)|^2 dx \right)^{\frac{1}{2}} \rightarrow 0 \quad (n \rightarrow \infty).$$

Ferner sei $u(x)$ diejenige Treppenfunktion, die gleich 1 für $0 \leq x \leq 1$, sonst gleich 0 ist,

$$(f, \phi) = \int_0^\infty f(x) \overline{\phi(x)} dx = \overline{(\phi, f)}; \quad \|f\| = (f, f)^{\frac{1}{2}} \geq 0,$$

die Mellinsche Transformierte Mf einer Funktion $f(x) \in \mathfrak{L}^2$

$$F(t) = \lim_{n \rightarrow \infty} \int_{1/n}^{2^n} f(x) x^{-\frac{1}{2} + it} dx = Mf; \quad f = M^{-1}F.$$

Die Adjungierte einer Transformation T , $T^*\phi = \phi^*$, ist definiert durch alle von der Wahl von $f \in \mathfrak{D}$ unabhängigen Lösungspaare ϕ, ϕ^* der Gleichung

$$(Tf, \phi) = (f, \phi^*). \quad (1.3)$$

2. Die Transformationen $T[\chi]$

Sei $T[\chi]$ die Transformation $g = Tf$,

$$\int_0^u g(\xi) d\xi = \int_0^\infty \frac{\chi(tu)}{t} f(t) dt, \quad \frac{\chi(x)}{x} \in \mathfrak{L}^2. \quad (2.1)$$

† Cf. M. H. Stone, Linear Transformations in Hilbert space, *American Math. Soc. Coll. Publications*, 15 (1932); I. v. Neumann, *Math. Annalen*, 102 (1930), 49–131 ('Neumann I') und *Annals of Math.* (2), 33 (1932), 294–310 ('Neumann II').

Als 'Hauptklasse' werde die Gesamtheit aller $T[\chi]$ bezeichnet, deren Bereich \mathfrak{D} gleich \mathfrak{Q}^2 ist; $T[\chi]$ ist dann beschränkt (V. Satz 2). Herr Plancherel hat zur Kennzeichnung dieser Transformationen die Funktionalgleichung†

$$T\{f(\alpha x)|y\} = \alpha^{-1}T\{f(x)|\alpha^{-1}y\}, \quad \alpha > 0 \quad (2.2)$$

herangezogen und u.a. gezeigt, dass jede lineare und beschränkte Transformation mit dem Bereich \mathfrak{Q}^2 , die (2.2) erfüllt, die Gestalt (2.1) hat. Der Operator $T[\chi]$ ist ein Hermitescher dann und nur dann, wenn χ reell ist, und unterliegt dann natürlich allen Gesetzen Hermitescher Operatoren (s. Neumann I und II).

Die Transformation $T[\chi]$ hat folgende Eigenschaften:

A. Sie ist linear und abgeschlossen.

B. Für jedes $a > 0$ gehört $u(x/a)$ zu \mathfrak{D} , dem Bereiche von T ; \mathfrak{D} ist in \mathfrak{Q}^2 überall dicht.

C. Zu $T[\chi]$ existiert eine Adjungierte T^* , sie ist $T[\bar{\chi}]$, d.h. von der Form (2.1) mit $\bar{\chi}$ statt χ ; daher ist auch $(Tf_1, \bar{f}_2) = (Tf_2, \bar{f}_1)$.

Die Adjungierte von T^* ist T , also $(T^*)^* = T$. Der Bereich \mathfrak{D}^* von T^* besteht aus allen zu den Funktionen von \mathfrak{D} konjugiert komplexen Funktionen; entsprechendes gilt für die Wertevorräte \mathfrak{R}^* und \mathfrak{R} .

D. Dann und nur dann ist T beschränkt, d.h. $\|Tf\| \leq A\|f\|$ für alle $f \in \mathfrak{D}$, wenn T zur Hauptklasse‡ gehört.

E₁. Damit eine Transformation T_1 , deren Bereich \mathfrak{D}_1 ist, der Gleichung (2.1) genügt, ist hinreichend: (i) T_1 erfüllt für jede Funktion $f \in \mathfrak{D}_1$ die Gleichung (2.2), (ii) $u(x) \in \mathfrak{D}_1$, (iii) $u(x)$ gehört auch zum Bereiche \mathfrak{D}_1^* der auf Grund von (i) und (ii) eindeutig bestimmten Adjungierten T_1^* .

E₂. (iii) kann durch die beiden Bedingungen (iii)' T_1 linear und (iv) $\left| \int_0^1 T_1 f dx \right| \leq A\|f\|$ für $f \in \mathfrak{D}_1$ ersetzt werden.

E₃. Damit T_1 mit einer Transformation $T[\chi]$, welche durch (2.1) dargestellt wird, identisch ist, ist notwendig und hinreichend: T_1 erfüllt (2.2) und T_1 ist die kleinste lineare abgeschlossene Erweiterung einer Transformation T_0 , deren Bereich nur aus den Funktionen $u(x/a)$, $a > 0$ besteht.

† Betr. eine in Proc. Cambridge Phil. Soc. erscheinende Arbeit, deren Manuskript mir Herr Plancherel freundlichst übersandt hat. Für $Tf(x) = g(y)$ ist also $Tf(\alpha x) = \alpha^{-1}g(\alpha^{-1}y)$ für jedes positive α .

‡ Für diese ist notwendig und hinreichend die Beschränktheit von

$$\omega(t) = (\tfrac{1}{2} - it)M(\chi(x)/x),$$

V. Satz 2.

F. Es ist unmöglich, $T[\chi]$ in zweckmässiger Art über den Bereich \mathfrak{D} hinaus zu erweitern.[†]

G. Dann und nur dann ist T längentreu,[‡] wenn T zur 'Fourier-klasse' (d.h. T gehört zur Hauptklasse und $T[\chi]$ ist invers zu $T[\bar{\chi}]$, s. V. Einleitung) gehört.

Beweise.

A. Die Linearität ist ersichtlich, da aus $g_1 = Tf_1$, $g_2 = Tf_2$ und (2.1) sofort $T(a_1f_1 + a_2f_2) = a_1g_1 + a_2g_2$ für komplexe a_1, a_2 folgt. Die Abgeschlossenheit ist es gleichfalls, da für $f_n \rightarrow f$, $Tf_n \rightarrow g$ durch bekannte Sätze über Konvergenz im Quadratmittel aus (2.1) sofort $g = Tf$ folgt.

B. Aus (2.1) folgt durch die übliche leichte Rechnung, dass die formal gebildete Transformierte von $u(x/a)$ gleich $\chi(ax)/x$ ist und wegen $\chi(x)/x \in \Omega^2$ für jedes $a > 0$ zu Ω^2 gehört, also $u(x/a) \in \mathfrak{D}$. Da nun jede Treppenfunktion die Gestalt $\sum c_\nu u(x/a_\nu)$, $\nu = 1, 2, \dots, n$, a_ν reell, c_ν komplex, hat, gehört jede Treppenfunktion zu \mathfrak{D} . Da jede Funktion $f(x)$ aus Ω^2 sich durch eine Folge von Treppenfunktionen im Quadratmittel approximieren lässt, ist \mathfrak{D} in Ω^2 überall dicht.||

C. Daher ist die Adjungierte T^* von T eindeutig bestimmt.

Sei $g = Tf$, $\phi^* = T^*\phi$, $F(t) = Mf$, $\Phi(t) = M\phi$, u.s.w. Die Gleichung (s. 1.3)

$$(f, \phi^*) = (Tf, \phi), \quad f \in \mathfrak{D}, \quad \phi \in \mathfrak{D}^* \quad (2.3)$$

ist mit Rücksicht auf (3.1) (Abschn. 3) äquivalent mit

$$\int_{-\infty}^{\infty} F(t) \bar{\Phi}^*(t) dt = \int_{-\infty}^{\infty} G(t) \bar{\Phi}(t) dt = \int_{-\infty}^{\infty} \omega(-t) F(t) \bar{\Phi}(-t) dt,$$

und da die Funktionen $f(x)$ in Ω^2 , also $F(t)$ in $L^2(-\infty, \infty)$ überall dicht liegen, mit $\Phi^*(t) = \bar{\omega}(-t)\Phi(-t)$, diese Gleichung (s. 3.1) mit

$$T^*\phi = \phi^*(x) = T[\bar{\chi}]\phi.$$

Die Adjungierte von T^* ist wieder T , da zu $\bar{\chi}$ konjugiert komplex wieder χ ist. Die weiteren Aussagen von C folgen aus dem Umstand, dass $T^*\phi$ und $T\bar{\phi}$ konjugiert komplex sind.

[†] Cf. Neumann I, Def. 6, 7, 9; T kann so 'maximal', sogar 'hypermaximal' genannt werden.

[‡] Neumann I, Def. 6 und Satz 10; cf. Stone, l.c., Def. 2, 19 {'isometric', $(Tf, Tg) = (f, g)$ }.

§ E. C. Titchmarsh, *The Theory of Functions* (Oxford, 1932), 12, 53.

|| \mathfrak{D} oder bereits die Menge der Funktionen $u(x/a)$ 'bestimmt' oder 'spannt die abgeschlossene lineare Mannigfaltigkeit Ω^2 auf', Stone, l.c., Neumann I und II. Das ist gerade die notwendige und hinreichende Bedingung für die Existenz der Adjungierten, Stone, 2.6.

D. Für lineare Transformationen ist Beschränktheit in \mathfrak{D} mit Stetigkeit in \mathfrak{D} äquivalent (Stone, 2.21). Nun ist $T[\chi]$ nach A linear abgeschlossen, nach B der Bereich \mathfrak{D} überall dicht in \mathfrak{L}^2 . Wenn daher T in \mathfrak{D} beschränkt, d.h. stetig sein soll, muss $\mathfrak{D} = \mathfrak{L}^2$ sein; ist umgekehrt $\mathfrak{D} = \mathfrak{L}^2$, gehört also T zur Hauptklasse, so ist T beschränkt (Neumann II, Satz 12).

E₁. Aus $u(x) \in \mathfrak{D}_1$ folgt $u(x/a) \in \mathfrak{D}_1$ wegen (2.2), daher folgt wie in C die Existenz von T_1^* . T_1^* erfüllt ebenfalls die Gleichung (2.2), denn aus (2.3) und (2.2) ergibt sich für $g = T_1 f$, $\gamma = T_1^* \phi$

$$\begin{aligned}(T_1 f, \phi(\alpha x)) &= (g(x), \phi(\alpha x)) = (\alpha^{-1} g(\alpha^{-1} x), \phi(x)) \\ &= (T_1 f(\alpha x), \phi(x)) = (f(\alpha x), \gamma(x)) = (f(x), \alpha^{-1} \gamma(\alpha^{-1} x)), \\ T_1^* \phi(\alpha x) &= \alpha^{-1} \gamma(\alpha^{-1} x) = \alpha^{-1} T_1^* \{\phi(x) | \alpha^{-1} y\}.\end{aligned}$$

Wegen (iii) gehört also auch $u(x/a)$ zu \mathfrak{D}_1^* ; setzt man $T_1^* u(x)$ gleich $\bar{\chi}(x)/x$, so wird $T_1^* u(x/a)$ gleich $\bar{\chi}(\alpha x)/x$, wählt man nun† in (2.3) $\phi(x)$ gleich $u(x/a)$, $T_1^* \phi$ gleich $\bar{\chi}(\alpha x)/a$, so folgt die mit (2.1) gleichbedeutende Beziehung (2.11) $(T_1 f, u(x/a)) = (f, \bar{\chi}(\alpha x)/x)$.

Durch (2.11) wird eine Transformation T dargestellt, die im allgemeinen nicht mit T_1 identisch, sondern eine Erweiterung von T_1 ist, z.B. wenn T_1 nur für die Funktionen $u(x/a)$ definiert ist.

E₂. Sei T_1 linear, hieraus und aus (i) und (iv) folgt (iii): Für $T_1 u = \chi(x)/x$ ist

$$\left(T_1 u \left(\frac{x}{a}\right), u\right) = \left(\frac{\chi(ax)}{x}, u(x)\right) = \left(\frac{\chi(x)}{x}, u\left(\frac{x}{a}\right)\right) = \left(u\left(\frac{x}{a}\right), \frac{\bar{\chi}(x)}{x}\right).$$

Hieraus folgt durch Multiplizieren mit geeigneten Zahlen und Addieren

$$(T_1 f_n(x), u(x)) = (f_n(x), \bar{\chi}(x)/x) \quad (2.4)$$

für jede Treppenfunktion f_n . Sei jetzt $f(x)$ aus \mathfrak{D} , $f_n \rightarrow f$, so ist

$$|(T_1 f, u) - (T_1 f_n, u)| = |(T_1 (f - f_n), u)| \leq A \|f - f_n\| \rightarrow 0$$

wegen (iv). So folgt aus (2.4) durch bekannte Sätze für alle $f \in \mathfrak{D}_1$

$$(T_1 f, u) = (f, \bar{\chi}(x)/x),$$

also $u(x) \in \mathfrak{D}_1^*$, $T_1^* u = \bar{\chi}(x)/x$ mit Rücksicht auf (2.3).

E₃. Die Bedingungen sind notwendig. Sei $Tf = g$, $\alpha > 0$, so ist zufolge (2.1)

$$\begin{aligned}\int_0^\infty \frac{\chi(tu)}{t} f(\alpha t) dt &= \int_0^\infty \chi\left(t \frac{u}{\alpha}\right) f(t) \frac{dt}{t} = \int_0^{u/\alpha} g(\xi) d\xi = \int_0^u g\left(\frac{\xi}{\alpha}\right) \frac{d\xi}{\alpha}, \\ Tf(\alpha x) &= \alpha^{-1} g(\alpha^{-1} x).\end{aligned}$$

† Cf. S. Bochner, *Annals of Math.* 35 (1934), 111-15.

Sei T_0 die auf den Bereich $u(x/a)$ eingeschränkte Transformation T ; da T_0 ein Teil von T , also $T_0 u(x/a) = \chi(ax)/x$ ist, so ist zufolge (2.3) die Adjungierte T_0^* definiert durch

$$T_0^* \phi = \phi^*, \quad (\phi^*(x), u(x/a)) = (\phi(x), \chi(ax)/x), \quad a > 0,$$

ist also $T[\bar{\chi}]$, daher nach C gleich T^* . Nun ist aber die Adjungierte von T^* gleich T , die von T_0^* die kleinste lineare abgeschlossene Erweiterung von T_0 (Neumann II, Satz 2), diese muss also gleich T sein.

Die Bedingungen sind auch hinreichend, denn aus den Voraussetzungen folgt entsprechend, wenn man $T_1 u$ gleich $\chi(x)/x$ setzt:

$$T_0^* = T[\bar{\chi}] = T^*, \quad T_1 = (T_0^*)^* = (T^*)^* = T.$$

Für die Identität von T_1 mit (2.1) sind die Bedingungen (i) bis (iv) notwendig, aber nicht hinreichend. Das lässt sich mit Hilfe der hier benutzten Schlüsse† leicht zeigen, ebenso der Satz:

E₄. Damit eine Transformation T_1 mit einer Transformation $T[\chi]$ der Hauptklasse identisch ist, ist notwendig und hinreichend die Erfüllung von (i), dass ferner ihr Bereich Ω^2 , dass schliesslich (iii), oder dass (iii)' und (iv) erfüllt ist, oder dass T_1 linear abgeschlossen oder linear stetig ist.

F. Bei einer Fortsetzung von $T[\chi]$ über ϑ hinaus, etwa bei Erweiterung von T durch eine Transformation T_2 , wird man zweckmässig fordern, dass T^* auch zu T_2 adjungiert ist. Aus C aber folgt wegen $(T^*)^* = T$, dass T_2 dann mit T identisch sein muss.

G. Der Bereich der Fourierklasse ist Ω^2 , ferner ist

$$T^{-1} = T^* = T[\bar{\chi}],$$

die Klasse unitär (V. Einleitung) also erst recht längentreu. Ist umgekehrt T längentreu, also auch $\|Tf\| = \|f\|$, so ist T beschränkt und gehört nach D zur Hauptklasse ($\vartheta = \Omega^2$), also auch T^* ; da T^*f und $T\bar{f}$ konjugiert komplex sind, ist auch T^* längentreu. Für alle Funktionen f, ϕ, ψ aus Ω^2 ist daher

$$(Tf, T\phi) = (f, \phi), \quad (Tf, \psi) = (f, T^*\psi), \quad (2.51)$$

$$(T^*f, T^*\psi) = (f, \psi), \quad (T^*f, \phi) = (f, T\phi). \quad (2.52)$$

Durch Vergleich folgt hier für $T\phi = \psi$ aus (2.51) $\phi = T^*\psi$, für $T^*\psi = \phi$ aus (2.52) $\psi = T\phi$; also ist $(T^*)^{-1} = T$, $T^{-1} = T^* = T[\bar{\chi}]$.

† (iv) folgt durch Anwendung von Schwarz' Ungleichung auf die rechte Seite von (2.1) für $u = 1$.

3. Weitere Eigenschaften, insbesondere Umkehrbarkeit von $T[\chi]$

Sei $F(t) = Mf$, $G(t) = Mg$, $\omega(t)$ die zu $T[\chi]$ gehörige Funktion $(\frac{1}{2} - it)M(\chi(x)/x)$, so ist (2.1) ($f \in \mathfrak{L}^2$, $g \in \mathfrak{L}^2$, $\chi(x)/x \in \mathfrak{L}^2$) nach V. Hilfssatz 2 (ii) und (iii) äquivalent mit

$$G(t) = F(-t)\omega(t); \quad F(t), G(t), \omega(t)/(\frac{1}{2} - it) \in L^2(-\infty, \infty). \quad (3.1)$$

Der Bereich \mathfrak{D} von $T[\chi]$ ist also die Gesamtheit aller Funktionen $f \in \mathfrak{L}^2$, für die $F(-t)\omega(t)$ zu $L^2(-\infty, \infty)$ gehört. Folgender Satz gilt, dessen Beweis hier nicht ausgeführt wird:

C'. Wenn eine Transformation W zur Plancherel-Watson-Produktklasse bzw. Plancherel-Watson-Klasse gehört,† so bildet sie für jedes $T[\chi]$ der Art (2.1) den Bereich \mathfrak{D} ein-eindeutig und vollständig auf sich selbst bzw. auf \mathfrak{D}^* ab, ebenso den Vorrat \mathfrak{R} auf sich selbst bzw. auf \mathfrak{R}^* .

Ferner gelten die Sätze

H. Dann und nur dann ist $T[\chi]$ umkehrbar, wenn der Vorrat \mathfrak{R} in \mathfrak{L}^2 überall dicht ist.

J. Dann und nur dann ist $T[\chi]$ umkehrbar, wenn $\omega(t)$ fast überall von 0 verschieden ist.

K. Dann und nur dann ist $T[\chi]$ schlechthin umkehrbar, wenn $|\omega(t)| \geq c > 0$ ist. Die Umkehrung ist dann eine Transformation $T[\tilde{\chi}]$ der Hauptklasse, also beschränkt. Die Aussage $|\omega(t)| \geq c > 0$ ist äquivalent mit $\|Tf\| \geq c\|f\|$, $c > 0$, für alle $f \in \mathfrak{D}$.

L. Das Produkt von Transformationen der Hauptklasse ist dann und nur dann eine schlechthin umkehrbare Transformation, wenn jeder Faktor es ist.

Beweise.

H. Der Satz folgt sofort mit Rücksicht auf C aus einem allgemeineren: Sei $g = Sf$ eine lineare, abgeschlossene Transformation, deren Bereich in \mathfrak{L}^2 überall dicht ist — sodass die Adjungierte S^* existiert und die gleichen Eigenschaften hat, — dann ist eine notwendige und hinreichende Bedingung für die Umkehrbarkeit von S , dass der Vorrat von S^* in \mathfrak{L}^2 überall dicht ist. Da S linear ist, ist S dann und nur dann umkehrbar, wenn aus $Sf = 0$ folgt: $f = 0$. Andererseits ist eine Menge von Funktionen ϕ^* dann und nur dann überall dicht in \mathfrak{L}^2 , wenn, falls $(f, \phi^*) = 0$ für alle ϕ^* gilt, $f = 0$

† P.-W.-Klasse: Sowohl W wie W^{-1} gehören zur Hauptklasse von (2.1); WR definiert dann die P.-W.-Produktklasse (V. Satz 3 und 6A); $g = Rf$ bedeutet die Transformation $g(x) = x^{-1}f(x^{-1})$.

folgt. Sei nun $S^*\phi = \phi^*$; der allgemeine Satz ergibt sich, da folgende beiden Gleichungen äquivalent sind: $(f, \phi^*) = 0$ und $Sf = 0$, wegen $(f, \phi^*) = (Sf, \phi)$ (cf. Neumann II, Beweis von Satz 7).

J. Sei $\omega(t) = 0$ in einer Menge E der t vom Masze $|E| > 0$. Sei $F(-t)$ gleich Null ausserhalb von E , $F(-t) \in L^2(E)$, aber sonst willkürlich in E angenommen. Auf Grund von (3.1) ist dann $G(t)$ für alle diese F Null, also auch $g(x) = 0$ für alle den F entsprechenden f , T nicht umkehrbar.

Ist aber $\omega(t)$ von Null verschieden (d.h. bis auf eine Menge vom Masze Null), so hat die Gleichung $F(-t)\omega(t) = 0$ nur die Lösung $F = 0$; T ist also umkehrbar, da die Zuordnung $F \longleftrightarrow G$, also auch $f \longleftrightarrow g$, ein-eindeutig ist.

K. Wenn $\omega(t)$ positiv nach unten beschränkt ist, ist $\tilde{\omega}(t) = \omega^{-1}(-t)$ nach oben beschränkt, $F(t) = G(-t)\tilde{\omega}(t)$ und somit auch $f(x) \in \Omega^2$ für jedes $g(x) \in \Omega^2$ eindeutig festgelegt, T^{-1} gehört zur Hauptklasse (cf. V. Satz 2 u. 3).

Wenn jedem $g \in \Omega^2$ ein f und somit jedem $G \in L^2(-\infty, \infty)$ ein F eindeutig entsprechen soll, muss $G(t)\omega^{-1}(t)$ für jedes $G \in L^2(-\infty, \infty)$ zu $L^2(-\infty, \infty)$ gehören, also (cf. V. Beweis von Satz 2) ω^{-1} beschränkt sein. Folglich ist $|\omega| \geq c > 0$. Auf Grund der Längentreue von $(2\pi)^{1/2}Mf$ und nach (3.1) ist nun die Ungleichung

$$\int_{-\infty}^{\infty} |F(-t)\omega(t)|^2 dt \geq c^2 \int_{-\infty}^{\infty} |F(-t)|^2 dt, \quad c > 0, \quad F(t) \in L^2 \quad (3.2)$$

äquivalent mit der Aussage $\|Tf\| \geq c\|f\|$, $c > 0$, und (3.2) folgt aus $|\omega(t)| \geq c$, $c > 0$; umgekehrt folgt aus (3.2) die letzte Ungleichung durch passende Wahl von $F(t)$.

L. Man nehme etwa zwei Transformationen T_1 und T_2 , $T_1 f = g$, $T_2 g = h$, $Mh = H(t)$. Die Behauptung folgt mit Rücksicht auf K und die Beschränktheit von $\omega_1(t)$ und $\omega_2(t)$ unschwer aus der Gleichung (s. 3.1)

$$H(t) = G(-t)\omega_2(t) = F(t)\omega_1(-t)\omega_2(t).$$

4. Eine andere Klasse unbeschränkter Operatoren

Sei $U[\chi]$ die Transformation $g = Uf$

$$\int_0^v g(\xi) d\xi = \int_0^{\infty} \chi\left(\frac{v}{t}\right) f(t) dt, \quad \frac{\chi(x)}{x} \in \Omega^2, \quad \text{also auch} \quad \chi\left(\frac{1}{x}\right) \in \Omega^2, \quad (4.1)$$

ϑ_u bzw. ϑ_u^* die Bereiche von U bzw. U^* , \mathfrak{R}_u bzw. \mathfrak{R}_u^* die Vorräte. Seien ϑ_t bzw. \mathfrak{R}_t Bereich bzw. Vorrat von $T[\chi]$ (2.1), ϑ_t^* bzw. \mathfrak{R}_t^* von $T[\bar{\chi}]$, $Rf = x^{-1}f(x^{-1})$, $\omega(t)$ die zu $T[\chi]$ sowohl wie zu $U[\chi]$ gehörige Funktion. Dann ist $Uf = TRf$, $MUf = \omega(t)Mf$ (cf. V. Abschnitt 6); U^* ist gleich RT^* , da $R^* = R$ ist und

$$(Uf, g) = (TRf, g) = (Rf, T^*g) = (f, RT^*g).$$

Für die Operatoren $U[\chi]$ lassen sich sämtliche Sätze A bis L entsprechend beweisen mit Abänderung lediglich von C, C', E und G. Die Hauptklasse ($\vartheta_u = \mathfrak{Q}^2$) bildet eine kommutative Gruppe. Eine Transformation $g = Sf$ mit dem Bereich \mathfrak{Q}^2 gehört dann und nur dann zur Hauptklasse von (2.1) bzw. (4.1), wenn MSf/MRf^\dagger bzw. MSf/Mf eine von der Wahl von $f \in \mathfrak{Q}^2$ unabhängige Funktion ist. Anstelle von C, C', E und G treten:

$$C''. \quad \vartheta_u = \vartheta_u^* = \vartheta_t^*, \quad \mathfrak{R}_u = \mathfrak{R}_u^* = \mathfrak{R}_t, \quad (U^*)^* = U,$$

jede Transformation der P.-W.-Produktklasse bildet ϑ_u ein-eindeutig und vollständig auf sich selbst ab, ebenso \mathfrak{R}_u , ferner ist

$$(Uf_2(x), x^{-1}\bar{f}_1(x^{-1})) = (Uf_1(x), x^{-1}\bar{f}_2(x^{-1})) \quad \text{für } f_1, f_2 \in \vartheta_u,$$

U^* aus U bestimmt durch $U^* = U[\chi^*]$,

$$\int_0^x \chi^*\left(\frac{1}{v}\right) dv = x \int_0^{1/x} \bar{\chi}\left(\frac{1}{v}\right) dv.$$

E'_1 . Damit eine Transformation U der Gleichung (4.1) genügt, ist hinreichend, dass U die Bedingungen erfüllt (cf. Plancherel II):

- (i) wenn $Uf(x) = g(x)$, so ist $Uf(\alpha x) = g(\alpha x)$ für jedes positive α .
- (ii) $u(x)$ gehört zu ϑ_u . (iii) $u(x)$ gehört zu ϑ^* .

E'_2 . (ii) und (iii) zusammen können ersetzt werden durch: (ii)'

$$x^{-1}u(x^{-1}) \in \vartheta_u, \quad \text{(iii)'} \quad U \text{ linear, (iv) } \left| \int_0^1 Uf dx \right| \leq A \|f\| \quad \text{für } f \in \vartheta_u.$$

E_3 bleibt, nur sind (2.1) und (2.2) durch (4.1) und (4.2), T_1, T_0 u.s.w. durch U_1, U_0, \dots zu ersetzen.

G. $U[\chi]$ ist nur dann längentreu, wenn $T[\chi]$ zur Fourierklasse gehört.

\dagger Das soll bedeuten: es gibt eine Funktion $\omega(t)$ von der Art, dass für alle $f \in \mathfrak{Q}^2$: $MSf = \omega(t)MRf$ ist. Hieraus folgt, dass $\omega(t)$ in $(-\infty, \infty)$ beschränkt ist (wie in V. Satz 2).

5. Beispiele

1. Sei

$$\chi_\alpha(z) = 2^\alpha \int_0^z \xi^{1-\alpha} J_\nu(\xi) d\xi, \quad \Re(\nu) \geq -\frac{1}{2}, \quad -\frac{1}{2} < \alpha < \frac{1}{2}.$$

SATZ 1. Für $0 \leq \alpha < \frac{1}{2}$ ist $T[\chi_\alpha]$ eine beschränkte, für $-\frac{1}{2} < \alpha < 0$ eine unbeschränkte Transformation. Für $-\frac{1}{2} < \alpha \leq 0$ ist $T[\chi_\alpha]$ schlechthin umkehrbar, für $0 < \alpha < \frac{1}{2}$ nur umkehrbar; die Umkehrung von $T[\chi_\alpha]$ ist $T[\chi_{-\alpha}]$.

Im Falle $\alpha = 0$ ist T die Hankelsche Transformation T_0 und gehört zur 'Watson-Klasse' (V. Einleitung), $T_0 = T_0^{-1}$, für reelles $\nu \geq -1$ gehört T_0 auch noch zur Fourier-Klasse und ist für $\nu = \pm \frac{1}{2}$ die Fourier-sin bzw. cos-Transformation. Sei also jetzt

$$-\frac{1}{2} < \alpha < \frac{1}{2}, \quad \Re(\nu) \geq -\frac{1}{2}.$$

Es ist

$$|J_\nu(x)| \leq A x^{\Re(\nu)} \quad (x > 0),$$

$$|\chi_\alpha(x)| \leq A \int_0^x \xi^{\frac{1}{2}-\alpha+\Re(\nu)} d\xi \leq A' x^b, \quad b = \frac{3}{2} - \alpha + \Re(\nu) > \frac{1}{2}. \quad (5.1)$$

Aus der wohlbekannten asymptotischen Darstellung von $J_\nu(z)$ für $z \rightarrow \infty$ und aus

$$\int_1^x \xi^r \cos(\xi + c) d\xi = \begin{cases} O(1) & (r \leq 0, x \rightarrow \infty), \\ O(x^r) & (r \geq 0, x \rightarrow \infty) \end{cases}$$

und der entsprechenden sin-Formel folgt leicht

$$\chi_\alpha(x) = O(x^{|\alpha|}) \quad (x \rightarrow \infty). \quad (5.2)$$

Die Funktion $\chi(x)/x$ gehört wegen (5.1) und (5.2) zu \mathcal{Q}^2 , somit existiert der Operator $T[\chi]$. Mit Rücksicht auf (5.1) und (5.2) und V. Satz 3(i) ist die zu T gehörige Funktion

$$\omega_\alpha(t) = \lim_{b \rightarrow \infty} 2^\alpha \int_{1/b}^b x^{1-\alpha} J_\nu(x) x^{-\frac{1}{2}+it} dx$$

$$\omega_\alpha(t) = 2^{\frac{1}{2}} \Gamma\left\{\frac{1}{2}(1-\alpha+\nu+it)\right\} / \Gamma\left\{\frac{1}{2}(1+\alpha+\nu-it)\right\}. \quad (5.3)$$

Für $|t| \rightarrow \infty$ ist $|\omega(t)| \sim c|t|^{-\alpha}$ wegen einer bekannten Abschätzung von $\Gamma(\sigma+it)$.

Daher ist für $0 \leq \alpha < \frac{1}{2}$ die Funktion $\omega(t)$ beschränkt, T daher nach V. Satz 2 gleichfalls, T gehört zur Hauptklasse. Für

$$-\frac{1}{2} < \alpha < 0$$

† Der allgemeinere Fall $\Re(\nu) > -\frac{3}{2}$, $-\frac{1}{2} < \alpha < 1 + \Re(\nu)$ führt zu weniger einfachen, doch völlig übersehbaren Ergebnissen.

ist $\omega(t)$ nicht beschränkt, also auch T nicht, da T nach V. Satz 2 nicht zur Hauptklasse gehören kann.

Für $-\frac{1}{2} < \alpha \leq 0$ ist nach (5.3) $\omega(t)$ positiv nach unten beschränkt, also T nach Satz K schlechthin umkehrbar, für $0 < \alpha < \frac{1}{2}$ ist das nicht der Fall. Zufolge (5.3) ist $\omega_{-\alpha}(t)\omega_{\alpha}(-t) = 1$, also T_{α} die Umkehrung von $T_{-\alpha}$, $T_{-\alpha}$ die von T_{α} (s. Beweisgang von J, K, u. V. Satz 3).

Der Bereich von T ist im Falle $0 \leq \alpha < \frac{1}{2}$ gleich \mathfrak{L}^2 , im Falle $-\frac{1}{2} < \alpha < 0$ die Gesamtheit aller Funktionen

$$f(x) = M^{-1}\{G(-t)\omega_{-\alpha}(t)\} = \frac{1}{2\pi} \text{l.i.m.}_{n \rightarrow \infty} \int_{-n}^n G(-t)\omega_{-\alpha}(t)x^{-i-t} dt,$$

wobei G sämtliche Funktionen aus $L^2(-\infty, \infty)$ durchläuft.

2. $\chi_1(x) = 0$ bzw. $\log x$ für $x < 1$ bzw. $x > 1$, $\omega(t) = 1/(\frac{1}{2} - it)$.

$\chi_2(x) = x$ bzw. 1 für $x < 1$ bzw. $x > 1$, $\omega(t) = 1/(\frac{1}{2} + it)$.

Hier gehören T_1, T_2 zur Hauptklasse, sind nach Satz J umkehrbar, aber nicht schlechthin, die Umkehrung hat, mit Rücksicht auf (3.1), nicht die Form (2.1). Die Transformationen lassen sich schliesslich (V. Satz 2 A) auf die Gestalt bringen:

$$g_1(x) = \frac{1}{x} \int_0^x f\left(\frac{1}{t}\right) \frac{dt}{t}, \quad g_2\left(\frac{1}{x}\right) = \int_0^x f(t) dt, \dagger$$

aus der auch leicht die Form der Umkehrungen sich ergibt.

$$3. \quad \chi_{\alpha}(x) = 2^{2\alpha} \int_0^x \xi^{1-\alpha} \left\{ \frac{2}{\pi} K_0(\xi) - Y_0(\xi) \right\} d\xi \quad \left(-\frac{1}{2} < \alpha < \frac{1}{2} \right),$$

$$\omega_{\alpha}(t) = [2^{\alpha} \Gamma\{\frac{1}{4}(1-\alpha+it)\} / \Gamma\{\frac{1}{4}(1+\alpha-it)\}]^2.$$

Für $T[\chi_{\alpha}]$ gilt auch Satz 1. Sei nun

$$\psi_{\alpha}(x) = 2^{1+\alpha} \pi^{-\frac{1}{2}} \int_0^x \xi^{1-\alpha} \cos \frac{1}{2} \xi^2 d\xi,$$

$S[\psi_{\alpha}]$ die zugehörige Transformation der Form (2.1), so gilt für S

† Cf. Titchmarsh, l.c., 396, $\phi = Rg_2$, $\psi = Rg_1$. Es folgt auch leicht: Wenn $g(x)$ zu \mathfrak{L}^2 und iMg zu $\mathfrak{L}^2(-\infty, \infty)$ gehören, ist $g(x)$ einem Integral äquivalent, und zwar

$$g(x) = \int_x^{\infty} h(t) \frac{dt}{t}, \quad h(t) \subset \mathfrak{L}^2.$$

auch Satz 1, $T[\chi]$ entsteht aus S durch die Faltung $T = SRS$ (cf. V. Beispiel 1), $\omega_\alpha(t)$ ist das Quadrat von $(\frac{1}{2}-it)M(\psi_\alpha(x)/x)$.

$$4. \omega_\alpha(t) = \begin{cases} t^{-\alpha} & (t > 0) \\ 0 & (t < 0) \end{cases}, \quad \frac{\chi_\alpha(x)}{x} = M^{-1}\left(\frac{\omega_\alpha(t)}{\frac{1}{2}-it}\right), \quad -\frac{1}{2} < \alpha < \frac{1}{2}.$$

$T[\chi_\alpha]$ ist für $\alpha \geq 0$ beschränkt, für $\alpha < 0$ unbeschränkt; $T[\chi_\alpha]$ besitzt keine Umkehrung (s.J.). Setzt man nun

$$\omega(t) = \omega_\alpha(t) + \omega_{-\alpha}(-t) = |t|^{-\alpha \operatorname{sign} t}, \quad \frac{\chi(x)}{x} = M^{-1}\left(\frac{\omega(t)}{\frac{1}{2}-it}\right),$$

so ist der Operator $T[\chi]$ für $\alpha \neq 0$ unbeschränkt und seine eigene Umkehrung, wegen $\omega(t)\omega(-t) = 1$. Nur für $\alpha = 0$, also $T = R$, gehört T zur Watsonklasse, dann und nur dann auch zur Fourierklasse.

SUMMATION FORMULAE AND SELF-RECIPROCAL FUNCTIONS

By A. P. GUINAND (*Oxford*)

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1. Introduction

NUMEROUS papers have been written on various 'summation formulae' connecting sums of the type

$$\sum_{n=1}^{\infty} a_n f(n)$$

with corresponding sums of the type

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} f(x) K(nx) dx,$$

where $K(x)$ is a Fourier kernel.

Ferrar* has recently given a general theory of such summation formulae. His theory depends on the convergence theory of Mellin transforms.

In this paper I show that, by using a different method, the theory of mean convergence for transforms of functions of $L^2(0, \infty)$ can be applied to the problem. As one would expect, the resulting form of the summation formula is more symmetrical, and the conditions under which it holds are considerably simplified.

2. Formalities

The method is best explained by an example. It is known† that the function

$$\{[x] - x\}x^{-1}$$

is self-reciprocal with respect to the Fourier kernel $2\pi x^{\frac{1}{2}} J_{\frac{1}{2}}(2\pi x)$. That is,

$$2\pi \int_0^{\infty} \{[t] - t\} x^{\frac{1}{2}} t^{-\frac{1}{2}} J_{\frac{1}{2}}(2\pi xt) dt = \{[x] - x\}x^{-1}.$$

Further, it can readily be shown that, if $f(x)$ and $g(x)$ are a pair of

* W. L. Ferrar, *Compositio Mathematica*, 1 (1935), 344-60 and 4 (1937), 394-405.

† G. H. Hardy and E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 1 (1930), 196-231 (213).

functions connected by the equations

$$g(x) = 2 \int_0^{\infty} f(t) \cos 2\pi x t \, dt, \quad (2.1)$$

$$f(x) = 2 \int_0^{\infty} g(t) \cos 2\pi x t \, dt, \quad (2.11)$$

then $xf'(x)$ and $xg'(x)$ are formally a pair of transforms with respect to the kernel $2\pi x^{\frac{1}{2}} J_{\frac{1}{2}}(2\pi x)$. Hence, by the Parseval theorem for such transforms,

$$\int_0^{\infty} \{[x] - x\} x^{-1} f'(x) \, dx = \int_0^{\infty} \{[x] - x\} x^{-1} g'(x) \, dx.$$

Integrating by parts and assuming that the integrated terms vanish, we have

$$\int_0^{\infty} f(x) \, d\{[x] - x\} = \int_0^{\infty} g(x) \, d\{[x] - x\}.$$

That is,

$$\lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N f(n) - \int_0^N f(x) \, dx \right\} = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N g(n) - \int_0^N g(x) \, dx \right\}. \quad (2.2)$$

If the series and integrals concerned converge, this becomes, by (2.1) and (2.11)

$$\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(x) \, dx + 2 \sum_{n=1}^{\infty} \int_0^{\infty} f(x) \cos 2\pi n x \, dx.$$

This is the well-known Poisson summation formula.

3. The self-reciprocal function

I have shown in a previous paper* that there is a connexion between a certain class of self-reciprocal functions of the type

$$\left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} x^{-\frac{1}{2}(\beta+1)}$$

and a class of summation formulae of the type

$$\frac{1}{\Gamma(\alpha)} \sum_{1 \leq n \leq x} a_n (x-n)^{\alpha-1} = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \psi(0) + R_{\alpha-1}(x) + \sum_{n=1}^{\infty} a_n n^{1-\alpha-\beta} A_{\alpha}(nx).$$

A set of conditions was given covering both results. For our present purpose we only require the results concerning the self-reciprocal

* A. P. Guinand, *Proc. London Math. Soc.* (2), 43 (1937), 439-48.

function, and consequently we can omit a number of assumptions made in the previous paper.

THEOREM 1. *If (i) there is a sequence $\{a_n\}$ of real numbers and a positive β such that*

$$\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (s = \sigma + it) \quad (3.1)$$

is convergent for sufficiently large values of σ , and can be continued analytically into the region $\sigma \geq \frac{1}{2}\beta$,

(ii) there exists a function $R_0(x)$ which can be expressed as the sum of a finite number of terms of the form $ax^m(\log x)^n$, where $m \geq 0$, and n is a non-negative integer, and further*

$$\sum'_{1 \leq n \leq x} a_n - R_0(x) = O(x^\eta) \quad \text{as } x \rightarrow \infty, \quad (3.2)$$

$$R_0(x) = O(x^\theta) \quad \text{as } x \rightarrow 0, \quad (3.3)$$

where

$$\theta > \frac{1}{2}\beta > \eta, \quad (3.4)$$

then

$$\int_0^x F(y) dy = \int_0^\infty F(y) \frac{\chi_1(xy)}{y} dy, \quad (3.5)$$

where

$$F(x) = x^{-\frac{1}{2}(\beta+1)} \left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\}, \quad (3.6)$$

and

$$\chi_1(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{x^{\frac{1}{2}-it} (\frac{1}{2}\beta + it) \psi(\frac{1}{2}\beta - it)}{(\frac{1}{2} - it)(\frac{1}{2}\beta - it) \psi(\frac{1}{2}\beta + it)} dt. \quad (3.7)$$

Further, $\chi_1(x)$ is a Fourier kernel in Watson's sense.†

This theorem is proved in precisely the same way as Theorem 3 of the previous paper.

4. The connexion between the two classes of transforms

Put
$$A(s) = \frac{\psi\{\frac{1}{2}(\beta+1)-s\}}{\psi\{\frac{1}{2}(\beta-1)+s\}} \quad (4.1)$$

and
$$\chi(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T A(\frac{1}{2}-it) \frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it} dt. \quad (4.2)$$

Then
$$A(\frac{1}{2}+it)A(\frac{1}{2}-it) = 1, \quad |A(\frac{1}{2}+it)| = 1,$$

* It is known that such functions exist in most of the cases arising in practice.

† G. N. Watson, *Proc. London Math. Soc.* (2), 35 (1932), 156-99.

and $\chi(x)$ is a Fourier kernel in Watson's sense. Further

$$\chi_1(x) = \beta x^{-1(\beta-1)} \int_0^x \chi(u) u^{1(\beta-3)} du - \chi(x), \quad (4.3)$$

as in the previous paper.

Now suppose that $f(x)$ belongs to $L^2(0, \infty)$, and consequently has a transform $g(x)$ with respect to the kernel $\chi(x)$, and $g(x)$ also belongs to $L^2(0, \infty)$. That is,

$$\int_0^x g(y) dy = \int_0^\infty f(y) \frac{\chi(xy)}{y} dy, \quad (4.4)$$

$$\text{and} \quad \int_0^x f(y) dy = \int_0^\infty g(y) \frac{\chi(xy)}{y} dy.$$

However, (4.4) does not determine $g(x)$ uniquely, and we remain at liberty to make arbitrary changes in the values of $g(x)$ for any set of values of x of zero measure.

Suppose that $\mathfrak{F}(s)$ and $\mathfrak{G}(s)$ are the Mellin transforms* of $f(x)$ and $g(x)$. That is

$$\mathfrak{F}(s) = \text{l.i.m.}_{X \rightarrow \infty} \int_{1/X}^X f(x) x^{s-1} dx \quad (\sigma = \tfrac{1}{2}),$$

and similarly for $\mathfrak{G}(s)$.

By the Parseval theorem for Mellin transforms (4.4) gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathfrak{G}(\tfrac{1}{2} + it) \frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathfrak{F}(\tfrac{1}{2} - it) \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it} x^{\frac{1}{2}-it} dt.$$

$$\text{Hence} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\mathfrak{F}(\tfrac{1}{2} - it) A(\tfrac{1}{2} - it) - \mathfrak{G}(\tfrac{1}{2} + it)\} \frac{x^{\frac{1}{2}-it}}{\frac{1}{2}-it} dt = 0$$

and, by the Mellin inversion formula,

$$\mathfrak{G}(\tfrac{1}{2} + it) = \mathfrak{F}(\tfrac{1}{2} - it) A(\tfrac{1}{2} - it) \quad (4.5)$$

almost everywhere. That is

$$\mathfrak{G}(s) = \mathfrak{F}(1-s) A(1-s)$$

almost everywhere on $\sigma = \tfrac{1}{2}$.

Now suppose further that $f(x)$ is an integral, and that $xf'(x)$

* For the theory of Mellin transforms of functions of L^2 see E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 94-5.

belongs to $L^2(0, \infty)$. Then $xf'(x)$ has a Mellin transform $\mathfrak{F}_1(s)$ which belongs to $L^2(-\infty, \infty)$ along $\sigma = \frac{1}{2}$. Now

$$\int_0^\infty xf'(x)f(x) dx, \quad \int_0^\infty \{f(x)\}^2 dx$$

exist as L -integrals, and

$$\int_0^\infty xf'(x)f(x) dx = [x\{f(x)\}^2]_0^\infty - \int_0^\infty \{f(x)\}^2 dx - \int_0^\infty xf'(x)f(x) dx.$$

Since all the integrals concerned converge, it follows that the integrated term must approach definite limits as $x \rightarrow +0$ or $x \rightarrow \infty$. That is,

$$f(x) \sim Ax^{-1} \quad \text{as } x \rightarrow +0$$

and

$$f(x) \sim Bx^{-1} \quad \text{as } x \rightarrow \infty.$$

Since $f(x)$ belongs to $L^2(0, \infty)$, A and B must vanish, and we have

$$xf(x) \rightarrow 0 \quad \text{as } x \rightarrow +0 \quad \text{or } x \rightarrow \infty.$$

Hence

$$\begin{aligned} \mathfrak{F}_1(\tfrac{1}{2}+it) &= \text{l.i.m.}_{X \rightarrow \infty} \int_{1/X}^X xf'(x)x^{-1+it} dx \\ &= \text{l.i.m.}_{X \rightarrow \infty} \{[x^{1+it}f(x)]_{1/X}^X - (\tfrac{1}{2}+it) \int_{1/X}^X f(x)x^{-1+it} dx\} \\ &= -(\tfrac{1}{2}+it)\mathfrak{F}(\tfrac{1}{2}+it), \end{aligned}$$

$$\text{i.e.} \quad \mathfrak{F}_1(s) = -s\mathfrak{F}(s),$$

and consequently

$$\tfrac{1}{2}(\beta-1)f(x)-xf'(x), \quad \{\tfrac{1}{2}(\beta-1)+s\}\mathfrak{F}(s)$$

are a pair of Mellin transforms. Hence, by (4.5),

$$\begin{aligned} |[\tfrac{1}{2}(\beta-1)+s]\mathfrak{G}(s)| &= \left| \frac{\tfrac{1}{2}(\beta-1)+s}{\tfrac{1}{2}(\beta+1)-s} \{\tfrac{1}{2}(\beta+1)-s\}A(1-s)\mathfrak{F}(1-s) \right| \\ &= |[\tfrac{1}{2}(\beta+1)-s]\mathfrak{F}(1-s)| \end{aligned}$$

on $\sigma = \frac{1}{2}$, since $|A(\frac{1}{2}+it)| = 1$. Thus $\{\frac{1}{2}(\beta-1)+s\}\mathfrak{G}(s)$ belongs to $L^2(-\infty, \infty)$ along $\sigma = \frac{1}{2}$ and has a Mellin transform $G(x)$ which belongs to $L^2(0, \infty)$.

$$G(x) = \text{l.i.m.}_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T (\tfrac{1}{2}\beta+it)\mathfrak{G}(\tfrac{1}{2}+it)x^{-1-it} dt.$$

Further, if $x > 0$,

$$\begin{aligned} \int_x^\infty |G(u)|u^{-\frac{1}{2}(\beta+1)} du &\leq \left\{ \int_x^\infty |G(u)|^2 du \right\}^{\frac{1}{2}} \left\{ \int_x^\infty u^{-\beta-1} du \right\}^{\frac{1}{2}} \\ &\leq \left\{ \int_0^\infty |G(u)|^2 du \right\}^{\frac{1}{2}} x^{-\frac{1}{2}\beta-\frac{1}{2}}, \end{aligned}$$

and hence the integral converges absolutely. Thus,

$$\begin{aligned}
 x^{\frac{1}{2}(\beta-1)} \int_x^\infty G(u) u^{-\frac{1}{2}(\beta+1)} du \\
 &= x^{\frac{1}{2}(\beta-1)} \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_x^\infty u^{-\frac{1}{2}(\beta+1)} du \int_{-T}^T (\tfrac{1}{2}\beta + it) \mathfrak{G}(\tfrac{1}{2} + it) u^{-\frac{1}{2}-it} dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \mathfrak{G}(\tfrac{1}{2} + it) x^{-\frac{1}{2}-it} dt \\
 &= g(x)
 \end{aligned} \tag{4.6}$$

almost everywhere, and the inversion of the order of integration is justified by absolute convergence.

Since we can still make arbitrary changes in the values of $g(x)$ for a set of values of x of zero measure, we can now choose $g(x)$ satisfying (4.6) everywhere.

With this value $g(x)$ is the integral of its derivative, and

$$x^{-\frac{1}{2}(\beta-1)} g(x) = \int_x^\infty G(u) u^{-\frac{1}{2}(\beta+1)} du.$$

Hence, on differentiating and multiplying by $x^{\frac{1}{2}(\beta+1)}$,

$$G(x) = \tfrac{1}{2}(\beta-1)g(x) - xg'(x).$$

Since $g(x)$ and $G(x)$ belong to $L^2(0, \infty)$, it follows that $xg'(x)$ also belongs to $L^2(0, \infty)$ and $x^{\frac{1}{2}}g(x)$ approaches zero as x approaches zero or infinity.

Now, by the Parseval theorem for Mellin transforms of L^2 and (4.5),

$$\begin{aligned}
 \int_0^\infty \{ \tfrac{1}{2}(\beta-1)f(y) - yg'(y) \} \frac{\chi_1(xy)}{y} dy \\
 &= \frac{1}{2\pi} \int_{-\infty}^\infty (\tfrac{1}{2}\beta + it) \mathfrak{F}(\tfrac{1}{2} + it) \frac{A(\tfrac{1}{2} + it)}{\tfrac{1}{2} + it} \frac{\tfrac{1}{2}\beta - it}{\tfrac{1}{2}\beta + it} x^{\frac{1}{2}+it} dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^\infty (\tfrac{1}{2}\beta - it) \mathfrak{G}(\tfrac{1}{2} - it) \frac{x^{\frac{1}{2}+it}}{\tfrac{1}{2} + it} dt \\
 &= \int_0^x \{ \tfrac{1}{2}(\beta-1)g(y) - yg'(y) \} dy.
 \end{aligned}$$

Thus we have proved

LEMMA α . If $f(x)$ belongs to $L^2(0, \infty)$ and is an integral, and $xf'(x)$ belongs to $L^2(0, \infty)$, then $f(x)$ has a transform $g(x)$ of $L^2(0, \infty)$ with respect to the kernel $\chi(x)$. Further, $g(x)$ is almost everywhere differentiable and can be chosen equal to the integral of its derivative, and $xg'(x)$ belongs to $L^2(0, \infty)$. Also $x^{\frac{1}{2}}f(x)$ and $x^{\frac{1}{2}}g(x)$ approach zero as x approaches zero or infinity, and

$$\frac{1}{2}(\beta-1)f(x)-xf'(x), \quad \frac{1}{2}(\beta-1)g(x)-xg'(x)$$

are a pair of transforms of $L^2(0, \infty)$ with respect to the kernel $\chi_1(x)$ given by (4.2).

5. The summation formula

By the Parseval theorem for χ_1 -transforms of $L^2(0, \infty)$

$$\begin{aligned} \int_0^\infty \left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} x^{-\frac{1}{2}(\beta+1)} \left\{ \frac{1}{2}(\beta-1)f(x) - xf'(x) \right\} dx \\ = \int_0^\infty \left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} x^{-\frac{1}{2}(\beta+1)} \left\{ \frac{1}{2}(\beta-1)g(x) - xg'(x) \right\} dx. \quad (5.1) \end{aligned}$$

The left-hand side is

$$\begin{aligned} - \int_0^\infty \left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} \frac{d}{dx} \{ x^{-\frac{1}{2}(\beta-1)} f(x) \} dx \\ = - \left[\left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} x^{-\frac{1}{2}(\beta-1)} f(x) \right]_0^\infty + \int_0^\infty x^{-\frac{1}{2}(\beta-1)} f(x) d \left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} \\ = [O\{x^{\frac{1}{2}}f(x)\}]_0^\infty + \lim_{N \rightarrow \infty} \int_0^N x^{-\frac{1}{2}(\beta-1)} f(x) d \left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} \\ = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N a_n n^{-\frac{1}{2}(\beta-1)} f(n) - \int_0^N x^{-\frac{1}{2}(\beta-1)} f(x) dR_0(x) \right\}. \end{aligned}$$

Treating the right-hand side of (5.1) in the same way, we have

THEOREM 2. If, in addition to the assumptions (i) and (ii) of Theorem 1,

(iii) $f(x)$ is an integral and $f(x)$ and $xf'(x)$ belong to $L^2(0, \infty)$, then

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N a_n n^{-\frac{1}{2}(\beta-1)} f(n) - \int_0^N x^{-\frac{1}{2}(\beta-1)} f(x) dR_0(x) \right\} \\ = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N a_n n^{-\frac{1}{2}(\beta-1)} g(n) - \int_0^N x^{-\frac{1}{2}(\beta-1)} g(x) dR_0(x) \right\}, \quad (5.2) \end{aligned}$$

where

$$\int_0^x g(y) dy = \int_0^\infty f(y) \frac{\chi(xy)}{y} dy,$$

$$\chi(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T A\left(\frac{1}{2} - it\right) \frac{x^{\frac{1}{2} - it}}{\frac{1}{2} - it} dt,$$

and $g(x)$ is chosen so that it is the integral of its derivative.

6. Convergence theory

So far we have only been concerned with the theory of mean convergence for our self-reciprocal functions. However, it is known that the self-reciprocal formula (3.5) converges in the ordinary sense for a number of particular examples,* and that it is equivalent to the formula (5.2) with

$$f(x) = x^{\frac{1}{2}(\beta-1)} \quad (0 \leq x \leq y),$$

$$= 0 \quad (x > y).$$

If we make some further assumptions we can deduce convergence in the ordinary sense for the general case.

First we require the following lemma, which is the extension of a theorem for Fourier integrals given by Titchmarsh.†

LEMMA β . If (i) $\chi_1(x)$ is an integral and has almost everywhere a derivative $K_1(x)$ which belongs to L^2 over any finite range,

$$(ii) \quad B(x, y, \lambda) = \int_0^\lambda K_1(ux) K_1(uy) du \quad (6.1)$$

and $(x-y)B(x, y, \lambda)$ is bounded for all positive x, y, λ ,

$$(iii) \quad \lim_{\lambda \rightarrow \infty} \int_0^\lambda K_1(ua) \chi_1(ub) \frac{du}{u} = \begin{cases} 1 & (0 < a < b), \\ \frac{1}{2} & (a = b), \\ 0 & (0 < b < a), \end{cases} \quad (6.2)$$

$$(iv) \quad \int_{y-\delta}^{y+\delta} |B(x, y, \lambda)| dx < A$$

for some positive δ and all y and λ ,

* See, for instance, J. R. Wilton, *Proc. London Math. Soc.* (2), 29 (1927), 168-88.

† E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), Theorem 58.

(v) $f(x)$ belongs to $L^2(0, \infty)$ and is of bounded variation in some neighbourhood of $x = y$, then

$$\frac{1}{2}\{f(y+0)+f(y-0)\} = \lim_{\lambda \rightarrow \infty} \int_0^\lambda g(x)K_1(xy) dx, \quad (6.3)$$

where $g(x)$ is the χ_1 -transform of $f(x)$.

Now, by (6.1),

$$\begin{aligned} \int_a^b B(x, y, \lambda) dx &= \int_0^\lambda K_1(uy) du \int_a^b \chi_1'(ux) dx \\ &= \int_0^\lambda K_1(uy) \{\chi_1(ub) - \chi_1(ua)\} \frac{du}{u}, \end{aligned}$$

and hence, by (6.2),

$$\lim_{\lambda \rightarrow \infty} \int_a^b B(x, y, \lambda) dx = 0,$$

if y is not in the interval $a \leq y \leq b$, and

$$\lim_{\lambda \rightarrow \infty} \int_{y-\delta}^y B(x, y, \lambda) dx = \lim_{\lambda \rightarrow \infty} \int_y^{y+\delta} B(x, y, \lambda) dx = \frac{1}{2}.$$

Hence it follows, by the general convergence theorems of Hobson,* that, if $\phi(x)$ is summable in $(0, \infty)$,

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \phi(x)(x-y)B(x, y, \lambda) dx = 0,$$

and also that

$$\lim_{\lambda \rightarrow \infty} \int_{y-\delta}^{y+\delta} f(x)B(x, y, \lambda) dx = \frac{1}{2}\{f(y+0)+f(y-0)\}.$$

Now $(x-y)^{-1}f(x)$ belongs to $L(y+\delta, \infty)$, since

$$\begin{aligned} \int_{y+\delta}^\infty |(x-y)^{-1}f(x)| dx &\leq \left\{ \int_{y+\delta}^\infty |f(x)|^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{y+\delta}^\infty (x-y)^{-2} dx \right\}^{\frac{1}{2}} \\ &\leq \delta^{-\frac{1}{2}} \left\{ \int_0^\infty |f(x)|^2 dx \right\}^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\lim_{\lambda \rightarrow \infty} \int_{y+\delta}^\infty f(x)B(x, y, \lambda) dx = 0,$$

and, similarly,

$$\lim_{\lambda \rightarrow \infty} \int_0^{y-\delta} f(x)B(x, y, \lambda) dx = 0.$$

* E. W. Hobson, *The Theory of Functions of a Real Variable* (Cambridge, 1926), ii. 429 and 449.

Hence
$$\lim_{\lambda \rightarrow \infty} \int_0^{\infty} f(x)B(x, y, \lambda) dx = \frac{1}{2}\{f(y+0) + f(y-0)\}.$$

Now, by the Parseval theorem for χ_1 -transforms of functions of $L^2(0, \infty)$,

$$\int_0^{\lambda} g(x)K_1(xy) dx = \int_0^{\infty} f(x)B(x, y, \lambda) dx,$$

and (6.3) follows.

Now we can prove the following result:

THEOREM 3. *If, in addition to the assumptions of Theorem 1,*

- (i) $\chi_1(x)$ satisfies the assumptions (i), (ii), (iii), and (iv) of Lemma β ,
- (ii) $\chi(x)$ is everywhere differentiable, and

$$\chi'(x) = K(x) = O(x^{-1}) \quad \text{as } x \rightarrow 0 \quad (6.4)$$

$$H(x) = \int_0^x K(u)u^{1(\beta-1)} du = O(x^{1\beta}) \quad \text{as } x \rightarrow \infty, \quad (6.5)$$

then

$$\sum'_{1 \leq n \leq x} a_n - R_0(x) = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N a_n n^{-\beta} H(nx) - \int_0^{N+1} H(xy)y^{-\beta} dR_0(y) \right\}, \quad (6.6)$$

where \sum' indicates that, if x is an integer, the term in which $n = x$ is to be halved.

It follows from (4.3) that

$$K_1(x) = \beta x^{-1(\beta+1)} \int_0^x u^{1(\beta-1)} K(u) du - K(x).$$

Substituting this result and integrating by parts, we find that

$$\int_a^b K_1(xy)y^{-1(\beta+1)} dy = x^{-1(\beta+1)} \{a^{-\beta} H(xa) - b^{-\beta} H(xb)\}.$$

Now, by Theorem 1 and Lemma β ,

$$\begin{aligned} \left\{ \sum'_{1 \leq n \leq x} a_n - R_0(x) \right\} x^{-1(\beta+1)} &= \lim_{N \rightarrow \infty} \int_0^{N+1} \left\{ \sum'_{1 \leq n \leq y} a_n - R_0(y) \right\} y^{-1(\beta+1)} K_1(xy) dy \\ &= \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N (a_1 + a_2 + \dots + a_n) \int_n^{n+1} K_1(xy) y^{-1(\beta+1)} dy - \right. \\ &\quad \left. - \int_0^{N+1} R_0(y) y^{-1(\beta+1)} K_1(xy) dy \right\} \\ &= \lim_{N \rightarrow \infty} \left\{ x^{-1(\beta+1)} \sum_{n=1}^N (a_1 + a_2 + \dots + a_n) [n^{-\beta} H(nx) - (n+1)^{-\beta} H(nx+x)] + \right. \\ &\quad \left. + [R_0(y) x^{-1(\beta+1)} y^{-\beta} H(xy)]_0^{N+1} - x^{-1(\beta+1)} \int_0^{N+1} H(xy) y^{-\beta} dR_0(y) \right\}. \quad (6.7) \end{aligned}$$

Also, by (3.3) and (6.4)

$$\begin{aligned} R_0(y)y^{-\beta}H(xy) &= R_0(y)y^{-\beta} \int_0^{xy} K(u)u^{\frac{1}{2}(\beta-1)} du \\ &= O(y^{\theta-\frac{1}{2}\beta}) = O(y^\epsilon) \quad \text{as } y \rightarrow 0. \end{aligned}$$

Hence the right-hand side of (6.7) becomes

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\{ x^{-\frac{1}{2}(\beta+1)} \sum_{n=1}^N a_n n^{-\beta} H(nx) - x^{-\frac{1}{2}(\beta+1)} \int_0^{N+1} H(xy)y^{-\beta} dR_0(y) + \right. \\ \left. + x^{-\frac{1}{2}(\beta+1)}(N+1)^{-\beta} H(Nx+x) \left[R_0(N+1) - \sum_{n=1}^N a_n \right] \right\}. \end{aligned}$$

By (3.2) and (6.5) the last term is $O(N^{-\epsilon})$ as N approaches infinity, and (6.6) follows.

It is now obvious that we can justify a form of (5.2) for a function $f(x)$ such that $x^{\frac{1}{2}(\beta-1)}f(x)$ is a step-function having only a finite number of discontinuities and vanishing outside a finite range. In this case (5.2) is a linear combination of results of the form (6.6).^{*} Combining this with Theorem 2 we have:

THEOREM 4. *If the conditions of Theorems 1 and 3 are satisfied, and $f(x)$ can be expressed as*

$$f_1(x) + x^{-\frac{1}{2}(\beta-1)}f_2(x),$$

where $f_1(x)$ satisfies the conditions of Theorem 2, and $f_2(x)$ is a step-function having only a finite number of points of discontinuity, and vanishing outside a finite range; then

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N a_n n^{-\frac{1}{2}(\beta-1)} \frac{1}{2} [f(n+0) + f(n-0)] - \int_0^N x^{-\frac{1}{2}(\beta-1)} f(x) dR_0(x) \right\} \\ = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N a_n n^{-\frac{1}{2}(\beta-1)} g(n) - \int_0^N x^{-\frac{1}{2}(\beta-1)} g(x) dR_0(x) \right\}, \quad (6.8) \end{aligned}$$

where $g(x)$ is the χ -transform of $f(x)$, and is chosen so that it is the integral of its derivative.

7. Examples

So far as I am aware no summation formula has been previously discussed in the symmetrical form (6.8). However, we can readily verify that this form agrees with known formulae in any of the particular cases.

^{*} If we know enough about the nature of the convergence of (6.6), we can extend this argument to cover functions of bounded variation in any finite interval. See J. R. Wilton, *Quart. J. of Math.* (Oxford), 3 (1932), 26-32.

Also, it is not necessary for the truth of Theorem 4 that $f(x)$ should be bounded in the neighbourhood of the origin. Dixon and Ferrar* have given modifications of Voronoï's and Poisson's summation formulae for functions with logarithmic singularities at the origin.

(A) As an example, consider the case $a_n = r(n)$, where $r(n)$ is the number of solutions in integers of $x^2 + y^2 = n$. Then, with $\beta = 1$,

$$K(x) = \pi J_0(2\pi x^{\frac{1}{2}}), \quad R_0(x) = \pi x.$$

Let

$$f(x) = kf_1(x) + f_2(x)$$

where

$$f_1(x) = \begin{cases} \log x & (x \leq 1), \\ 0 & (x > 1), \end{cases}$$

and $f_2(x)$ is bounded in a neighbourhood of the origin, vanishes for $x > a$, and satisfies the conditions of Theorem 4.

Obviously $f_1(x)$ satisfies the conditions of Theorem 2, and (5.2) becomes

$$\pi = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N r(n)g_1(n) - \pi \int_0^N g_1(x) dx \right\} \quad (7.1)$$

where

$$g_1(x) = \pi \int_0^1 \log t J_0(2\pi x^{\frac{1}{2}} t^{\frac{1}{2}}) dt.$$

Thus

$$\begin{aligned} \pi \int_0^N g_1(x) dx &= \pi^2 \int_0^1 \log t dt \int_0^N J_0(2\pi x^{\frac{1}{2}} t^{\frac{1}{2}}) dx \\ &= 2 \int_0^{2\pi N^{\frac{1}{2}}} \log u J_1(u) du - \log 4\pi^2 N \int_0^{2\pi N^{\frac{1}{2}}} J_1(u) du. \end{aligned}$$

Now, if we differentiate

$$\int_0^\infty J_1(u) u^{s-1} du = 2^{s-1} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}s)}{\Gamma(\frac{3}{2} - \frac{1}{2}s)} \quad (-1 < s < \frac{3}{2}),$$

with respect to s , and put $s = 1$, we find that

$$\int_0^\infty \log u J_1(u) du = \log 2 - \gamma.$$

Hence

$$\pi \int_0^N g_1(x) dx = -2(\gamma + \log \pi) - \log N + O(N^{-\frac{1}{2}} \log N),$$

* A. L. Dixon and W. L. Ferrar, *Quart. J. of Math.* (Oxford), 8 (1937), 66-74.

and (7.1) becomes

$$\begin{aligned}\pi &= \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N r(n)g_1(n) + 2(\gamma + \log \pi) + \log N \right\} \\ &= \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N [r(n)g_1(n) + n^{-1}] + 2(\gamma + \log \pi) - \left[\sum_{n=1}^N n^{-1} - \log N \right] \right\} \\ &= \gamma + 2 \log \pi + \sum_{n=1}^{\infty} \{r(n)g_1(n) + n^{-1}\}.\end{aligned}\quad (7.2)$$

Now
$$g_1(0) = \pi \int_0^1 \log t \, dt = -\pi;$$

so we can write (7.2) as

$$0 = \sum_{n=1}^{\infty} r(n)f_1(n) = \gamma + 2 \log \pi + g_1(0) + \sum_{n=1}^{\infty} \{r(n)g_1(n) + n^{-1}\}. \quad (7.3)$$

Now consider $f_2(x)$. Put

$$g_2(x) = \pi \int_0^a f_2(t) J_0(2\pi x^{1/2} t) \, dt.$$

Then

$$g_2(0) = \pi \int_0^a f_2(t) \, dt,$$

and, by the inverse formula,

$$f_2(+0) = \pi \int_0^a g_2(t) \, dt.$$

Hence (6.8) becomes

$$f_2(+0) + \sum_{1 \leq n \leq a} r(n) \frac{1}{2} \{f_2(n+0) + f_2(n-0)\} = g_2(0) + \sum_{n=1}^{\infty} r(n)g_2(n).$$

Combining this with (7.3) we have:

THEOREM 5.* *If $f(x)$ is defined in $(0, a)$ and there exists a number k such that*

$$F(x) = f(x) - k \log x$$

is bounded in some neighbourhood of the origin, and $f(x)$ satisfies the conditions of Theorem 4, then

$$\begin{aligned}F(+0) + \sum_{1 \leq n \leq a} r(n) \frac{1}{2} \{f(n+0) + f(n-0)\} \\ = k(\gamma + 2 \log \pi) + g(0) + \sum_{n=1}^{\infty} \left\{ r(n)g(n) + \frac{k}{n} \right\}\end{aligned}$$

where

$$g(x) = \pi \int_0^a f(t) J_0(2\pi x^{1/2} t) \, dt.$$

* The corresponding result for the Poisson formula has been given by Dixon and Ferrar, loc. cit. They have asked me to point out the following mistake:

In the enunciation of Theorem 6 the second line of equation (6.21) should begin with the term $\frac{1}{2}b \log 2\pi$, not $\frac{1}{2}b(\gamma - \log 2)$.

(B) Another example in which the series on the right of (6.8) does not converge is obtained by taking $a_n = r_3(n)$ in Theorem 3. We find that

$$\sum_{n \leq x}' r_3(n) - \frac{4}{3}\pi x^{\frac{1}{2}} = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N r_3(n) \left(\frac{x}{n} \right)^{\frac{3}{2}} J_{\frac{3}{2}}(2\pi n^{\frac{1}{2}} x^{\frac{1}{2}}) + \frac{2}{\pi} \sin 2\pi N^{\frac{1}{2}} x^{\frac{1}{2}} \right\}.$$

If we investigate the summability of the series

$$\sum_{n=1}^{\infty} \{ \sin 2\pi(n+1)^{\frac{1}{2}} x^{\frac{1}{2}} - \sin 2\pi n^{\frac{1}{2}} x^{\frac{1}{2}} \},$$

we can readily deduce that

$$\sum_{n \leq x}' r_3(n) - \frac{4}{3}\pi x^{\frac{1}{2}} = \sum_{n=1}^{\infty} r_3(n) \left(\frac{x}{n} \right)^{\frac{3}{2}} J_{\frac{3}{2}}(2\pi n^{\frac{1}{2}} x^{\frac{1}{2}}),$$

where the series on the right is summable (R, n, ϵ) for any positive ϵ , but does not converge.*

(C) An example which does not come directly under our theorems, but which may be justified by using the theory of transforms of functions of $L^p(0, \infty)$ ($1 < p \leq 2$), is obtained by putting

$$f(x) = x^{-s} \quad \left(\frac{1}{2} < s < 1 \right)$$

in (2.2). We find that

$$\lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n^{-s} - \frac{N^{1-s}}{1-s} \right\} = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1-s) \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n^{s-1} - \frac{N^s}{s} \right\}.$$

The expressions in brackets give the analytic continuation of $\zeta(s)$,† and we have

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{1}{2} s \pi \Gamma(1-s) \zeta(1-s),$$

the functional equation of the zeta-function.

(D) Another example of a different type arises from the Dirichlet's series‡

$$\begin{aligned} \eta^2(s) &= \left\{ \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} \right\}^2 \\ &= \sum_{n=0}^{\infty} (-1)^n d(2n+1) (2n+1)^{-s} \end{aligned}$$

where $d(n)$ is the number of divisors of n .

* Compare A. Oppenheim, *Proc. London Math. Soc.* (2), 26 (1927), 295–350.

† E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 3.

‡ It can be shown that

$$\sum_{0 \leq 2n+1 \leq x} (-1)^n d(2n+1) = O(x^{27/82} \log x)$$

by the method of E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 2 (1931), 161–73, and hence the inequalities (3.2), (3.3), and (3.4) are satisfied.

With $\beta = 1$, we have

$$\chi(x) = -x^{\frac{1}{2}}M_1(\pi x^{\frac{1}{2}}), \quad K(x) = -\frac{1}{2}\pi L_0(\pi x^{\frac{1}{2}}), \quad R_0(x) = 0,$$

where

$$L_0(z) = -Y_0(z) - \frac{2}{\pi}K_0(z),$$

$$M_1(z) = -Y_1(z) + \frac{2}{\pi}K_1(z).$$

Hence the integrals in (6.8) vanish, and it follows that the series converge. Thus we have

THEOREM 6. *If $f(x)$ is an integral, and $f(x)$ and $xf'(x)$ belong to $L^2(0, \infty)$, then*

$$\sum_{n=0}^{\infty} (-1)^n d(2n+1)f(2n+1) = \sum_{n=0}^{\infty} (-1)^n d(2n+1)g(2n+1)$$

where
$$\int_0^x g(y) dy = - \int_0^{\infty} f(y) \left(\frac{x}{y}\right)^{\frac{1}{2}} M_1(\pi x^{\frac{1}{2}} y^{\frac{1}{2}}) dy,$$

and $g(x)$ is chosen so that it is the integral of its derivative.

Similar results hold for the summation formula

$$\sum_{n=0}^{\infty} (-1)^n f(2n+1) = \sum_{n=0}^{\infty} (-1)^n g(2n+1),$$

where $f(x)$ and $g(x)$ are Fourier sine transforms with respect to the kernel $\sin \frac{1}{2}\pi x$, and also for a large class of summation formulae arising from other Dirichlet L -functions and their squares.

1 SOME RESULTS IN THE ADDITIVE PRIME- NUMBER THEORY

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SINCE Siegel† and Vinogradow‡ solved the essential difficulties, a great many theorems in the additive prime-number theory can be proved without any hypothesis (formerly the Riemann hypothesis was used to deal with such problems). The object of the present note§ is to prove the following results:

THEOREM 1. *Let A_k be the set of all those integers n which satisfy the conditions*

(i) *if k is odd*

$$n \not\equiv 0 \pmod{2}, \quad n \not\equiv 2 \pmod{3};$$

(ii) *if k is even*

$$n \equiv 3 \pmod{24}, \quad \begin{cases} n \not\equiv 0 \pmod{5} \text{ for } 4 \nmid k, \\ n \not\equiv 0, 2 \pmod{5} \text{ for } 4 \mid k, \end{cases}$$

and $n \not\equiv 1 \pmod{p}$ for $p \equiv 3 \pmod{4}$ and $(p-1) \mid k$.

Then almost all integers of the set A_k are sums of two squares of primes and a k th power of a prime.

CONSEQUENCES: 1. *Almost all integers congruent to 3 (mod 24) and not congruent to 0 (mod 5) are sums of three squares of primes.*

2. *Almost all integers congruent to 1 or 3 (mod 6) are sums of two squares of primes and a cube of a prime.*

THEOREM 2. *Every sufficiently large even integer is a sum of a prime and two squares of primes and a k th power of a prime for odd k . Every sufficiently large even integer not congruent to 0 (mod 3) is a sum of a prime and two squares of primes and a k th power of a prime for even k .*

By the same method I have also proved that

(i) almost all even integers are sums of one prime and a k th power of a prime for odd k ;

(ii) almost all even integers not congruent to 1 (mod p), for all those primes p satisfying $(p-1) \mid k$, are sums of one prime and a k th power of a prime for even k ;

† *Acta Arith.* 1 (1936), 83-6.

‡ *Comptes Rendus de l'URSS*, 16 (1937), no. 3.

§ A preliminary account of this paper has appeared in *Comptes Rendus de l'URSS*, 18 (1938), No. 1.

(iii) all sufficiently large odd integers are sums of two primes and a k th power of a prime;

(iv) all sufficiently large integers congruent to 5 (mod 24) are sums of four squares of primes and a k th power of a prime, if k is even;

(v) all sufficiently large odd integers not congruent to 1 (mod 3) are sums of four squares of primes and a k th power of a prime, if k is odd.

The proofs of them are very similar to those given in this note, so that they need not be published. In the proof of Theorem 1 I shall use a method due to Davenport and Heilbronn† for the singular series.

Let N be a large integer and $L = \log N$ and $P^k = N$. Throughout the paper all small italic letters except c , e , and i denote positive integers, and p always denotes a prime; c_1, c_2, \dots are positive numbers depending only on k . The constants implied by the symbol O depend only on k and ϵ .

We divide the interval $0 \leq \alpha \leq 1$ in the usual way into Farey arcs corresponding to all Farey fractions a/q with

$$q \leq NL^{-h_0}, \quad a \leq q, \quad (a, q) = 1,$$

where h_0 is an integer to be fixed later (cf. Lemma 2). We divide these arcs into major arcs \mathfrak{M} ($q \leq L^{h_0}$) and minor arcs \mathfrak{m} ($L^{h_0} \leq q \leq NL^{-h_0}$). In either case an arc has the form

$$\alpha = a/q + \beta, \quad -\vartheta_1 q^{-1} N^{-1} L^{h_0} \leq \beta \leq \vartheta_2 q^{-1} N^{-1} L^{h_0},$$

where $\frac{1}{2} \leq \vartheta_1 \leq 1$, $\frac{1}{2} \leq \vartheta_2 \leq 1$. We use the abbreviation

$$e(\eta) = e^{2\pi i \eta}, \quad e_q(\eta) = e(\eta/q).$$

Let

$$V_k(\alpha) = \sum_{p^k \leq N} e(\alpha p^k), \quad S_{a,q}^{(k)} = \sum_{\substack{l=1 \\ (l,q)=1}}^q e_q(al^k),$$

$$V_k^*(\alpha, a, q) = \{\phi(q)\}^{-1} S_{a,q}^{(k)} \sum_{2 \leq n \leq N} n^{1/k-1} (\log n)^{-1} e(n\beta).$$

1. Preliminary lemmas

LEMMA 1 (Siegel-Walfisz).‡ If $q \leq L^{h_0}$, $(l, q) = 1$, $n \leq N$, then

$$\pi(n; q, l) = \{\phi(q)\}^{-1} \text{li } n + O(Ne^{-c_1 \sqrt{L}}).$$

LEMMA 2 (Vinogradow). For a given h there exists an integer h_0 such that on \mathfrak{m}

$$V_k(\alpha) = O(PL^{-h}).$$

† Proc. London Math. Soc. 43 (1936), 73–104.

‡ Math. Zeits. 40 (1936), 598, Hilfssatz 3.

LEMMA 3 (Sierpiński).† Let $R_2(n)$ be the number of solutions of $u = x^2 + y^2$. Then

$$\sum_{n=1}^N R_2^2(n) = O(NL).$$

2. Lemmas in general

Let $p^\theta \mid k$, $p^{\theta+1} \nmid k$, and

$$\gamma = \begin{cases} \theta+2 & \text{if } p = 2, \\ \theta+1 & \text{if } p \neq 2. \end{cases}$$

LEMMA 4. If $t > \gamma$, then

$$S_{a,p^t}^{(k)} = 0.$$

Proof. Let $l = l_1 + l_2 p^{t-\theta-1}$. By Landau,‡ Satz 290, we have

$$S_{a,p^t}^{(k)} = \sum_{\substack{l_1=1 \\ (l_1,p)=1}}^{p^{t-\theta-1}} \sum_{l_2=1}^{p^{\theta+1}} e_{p^t}\{a(l_1^k + p^{t-\theta-1} l_1^k l_2)\} = 0$$

since $p \nmid l_1 k p^{-\theta}$.

LEMMA 5. If $k \geq 2$, then

$$S_{a,q}^{(k)} = O(q^{k+\epsilon}).$$

If $k = 1$, evidently $S_{a,q}^{(1)} = \mu(q) = O(1)$,

where $\mu(q)$ is Möbius's function.

Proof. It is easy to verify that, if $(q_1, q_2) = 1$, then

$$S_{a,q_1 q_2}^{(k)} = S_{a q_1^{k-1}, q_2}^{(k)} S_{a q_2^{k-1}, q_1}^{(k)}.$$

By Lemma 4 we have

$$S_{a,p^t}^{(k)} = O(1) \quad \text{for } p \mid k \text{ or } p = 2, t = 1;$$

$$S_{a,p^t}^{(k)} = 0 \quad \text{for } p \nmid k \text{ and } p \neq 2, t > 1 \text{ or } p = 2, t > 2.$$

Further, by Landau‡ (Satz 311), we have

$$\begin{aligned} |S_{a,p}^{(k)}| &\leq k\sqrt{p} \quad \text{for all } p, \\ &\leq p^{k+\epsilon} \quad \text{for } p \geq k^{1/\epsilon}. \end{aligned}$$

Let $q = p_1^{t_1} \dots p_v^{t_v}$ and $p_1 < p_2 < \dots < p_v$. Then

$$S_{a,q}^{(k)} = \prod_{i=1}^v S_{a_i p_i^{t_i}}^{(k)},$$

where the a_i are integers satisfying $(a_i, p_i) = 1$. Consequently, we have

$$S_{a,q}^{(k)} = \prod_{p_i \leq k^{1/\epsilon}} |S_{a_i p_i^{t_i}}^{(k)}| \prod_{p_i > k^{1/\epsilon}} |S_{a_i p_i^{t_i}}^{(k)}| = O(q^{k+\epsilon}).$$

† *Prac. mat.-fiz.* 18 (1907), 1-60 (Polish). I am indebted to Prof. Walfisz for this reference.

‡ *Vorlesungen über Zahlentheorie*, I (1927). I shall not repeat this footnote on similar occasions.

LEMMA 6. On \mathfrak{M}

$$V_k(\alpha) - V_k^*(\alpha, a, q) = O(Pe^{-c_3\sqrt{L}}).$$

Proof. Let $S_n = \sum_{p^k \leq n} e_q(ap^k), \quad n \leq N.$

Then, by Lemma 1, we have

$$\begin{aligned} S_n &= \sum_{\substack{l=1 \\ (l,q)=1}}^q e_q(al^k) \pi(n^{1/k}, q, l) + O(q^\epsilon) \\ &= \sum_{\substack{l=1 \\ (l,q)=1}}^q e_q(al^k) \left(\frac{1}{\phi(q)} \operatorname{li} n^{1/k} + O(Pe^{-c_1\sqrt{L}}) \right) + O(q^\epsilon) \\ &= \frac{S_{a,q}^{(k)}}{\phi(q)} \operatorname{li} n^{1/k} + O(Pe^{-c_3\sqrt{L}}). \end{aligned}$$

Hence

$$\begin{aligned} V_k(\alpha) &= \sum_{n=2}^N (S_n - S_{n-1}) e(n\beta) \\ &= \sum_{n=2}^N S_n \{e(n\beta) - e((n+1)\beta)\} + S_N e((N+1)\beta) \\ &= \frac{S_{a,q}^{(k)}}{\phi(q)} \left(\sum_{n=2}^N \operatorname{li} n^{1/k} \{e(n\beta) - e((n+1)\beta)\} + \operatorname{li} N^{1/k} e((N+1)\beta) \right) + \\ &\quad + O(Pe^{-c_3\sqrt{L}}), \end{aligned}$$

whence the result, since

$$\operatorname{li} n^{1/k} - \operatorname{li}(n-1)^{1/k} = \frac{1}{n^{1-1/k} \log n} + O\left(\frac{1}{n^{2-1/k} \log n}\right).$$

LEMMA 7. If $|\beta| \leq \frac{1}{2}$, then

$$\begin{aligned} V_k^*(\alpha, a, q) &= O\left(\frac{q^{\frac{1}{2}+\epsilon}}{\phi(q)} \min(PL^{-1}, |\beta|^{-1/k})\right) \quad \text{for } k > 1, \\ &= O\left(\frac{1}{\phi(q)} \min(NL^{-1}, |\beta|^{-1})\right) \quad \text{for } k = 1. \end{aligned}$$

Proof. By Lemma 5 we have

$$V_k^*(\alpha, a, q) = O\left(\frac{q^{\frac{1}{2}+\epsilon}}{\phi(q)} PL^{-1}\right).$$

The first inequality follows at once on dividing the sum into $n \leq |\beta|^{-1}$ and $n > |\beta|^{-1}$ and applying Abel's lemma to the second part. The second formula in the lemma can be proved in the same way.

3. Proof of Theorem 1

Let $k \geq 2$,† and

$$V_2^2(\alpha)V_k(\alpha) = \sum_n r(n)e(n\alpha),$$

so that, for $n \leq N$, $r(n)$ is the number of representations of n as $p_1^2 + p_2^2 + p_3^k$. Let

$$A_q(n) = \frac{1}{\phi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^q (S_{a,q}^{(2)})^2 S_{a,q}^{(k)}(-na),$$

$$\mathfrak{S}(n) = \sum_{q \leq L^7} A_q(n),$$

$$\mathfrak{S}_1(n) = \{1 + A_2(n) + A_{2^2}(n) + A_{2^3}(n)\} \prod_{2 < p < \frac{1}{2}L} \{1 + A_p(n)\}.$$

Choose h_0 so that the h of Lemma 2 is greater than or equal to 4. Let $M_{a,q}$ be the set of intervals with

$$q \leq L^7, \quad |\beta| \leq P^{-k}L^{6k},$$

and E the set of those points of the interval $(0, 1)$ which do not belong to any $M_{a,q}$.

LEMMA 8.
$$\int_E |V_2^2(\alpha)V_k(\alpha)|^2 d\alpha = O(P^{k+2}L^{-7}).$$

Proof. If α is on m , then, by Lemma 2,

$$V_k(\alpha) = O(PL^{-4}).$$

If $q \geq L^7$, then, by Lemmas 6 and 7,

$$V_k(\alpha) = O(Pe^{-c_2\sqrt{L}}) + O(q^{-\frac{1}{2}+\epsilon}PL^{-1}) = O(PL^{-\frac{1}{2}+\epsilon}).$$

If $|\beta| \geq P^{-k}L^{6k}$, then, by Lemmas 6 and 7,

$$V_k(\alpha) = O(Pe^{-c_2\sqrt{L}}) + O(q^{-\frac{1}{2}+\epsilon}|\beta|^{-1/k}) = O(PL^{-6}).$$

Therefore, by Lemma 3,

$$\int_E |V_k(\alpha)V_2^2(\alpha)|^2 d\alpha = O\left(P^2L^{-8} \int_0^1 |V_2(\alpha)|^4 d\alpha\right) = O(P^{k+2}L^{-7}).$$

LEMMA 9.

$$\sum_{q \leq L^7} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{M_{a,q}} |V_2^2(\alpha)V_k(\alpha) - V_2^{*2}(\alpha, a, q)V_k^*(\alpha, a, q)|^2 d\alpha = O(P^{k+2}e^{-c_3\sqrt{L}}).$$

Proof. By Lemmas 6 and 7,

$$\begin{aligned} |V_2^2V_k - V_2^{*2}V_k^*| &\leq |V_2^2 - V_2^{*2}||V_k| + |V_k - V_k^*||V_2^*|^2 \\ &\leq |V_2 - V_2^*||V_k|(|V_2| + |V_2^*|) + |V_k - V_k^*||V_2^*|^2 \\ &= O(P^{k+1}e^{-c_4\sqrt{L}}). \end{aligned}$$

† If $k = 1$, the problem is much easier; compare the result (i) in the introduction.

Therefore the expression in question is

$$O\left(\sum_{q \leq L'} q \int_0^{P^{-k}L^{6k}} P^{2(k+1)} e^{-2c_s \sqrt{L}} d\beta\right) = O(P^{k+2} e^{-c_s \sqrt{L}}).$$

LEMMA 10.

$$\int_0^1 \left| \sum_{q \leq L'} \sum_{\substack{a=1 \\ (a,q)=1}}^q V_2^{*2}(\alpha, a, q) V_k^*(\alpha, a, q) \right|^2 d\alpha = O(P^{k+2} L^{-7}),$$

where \sum' denotes that, if α is on one of the arcs $M_{a,q}$, the term a, q corresponding to this arc is to be omitted from the sum.

Proof. By Cauchy's inequality

$$\left| \sum_{q \leq L'} \sum_{\substack{a=1 \\ (a,q)=1}}^q V_2^{*2} V_k^* \right|^2 \leq \left(\sum_q \sum_a 1 \right) \left(\sum_q \sum_a |V_2^{*2} V_k^*|^2 \right).$$

Hence the integral in question does not exceed

$$\begin{aligned} L^{14} \sum_{q \leq L'} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(\int_0^1 - \int_{M_{a,q}} \right) |V_2^{*2}(\alpha, a, q) V_k^*(\alpha, a, q)|^2 d\alpha \\ = O\left(L^{14} \sum_{q \leq L'} \phi(q) \int_{P^{-k}L^{6k}}^{\infty} \frac{q^{3+\epsilon}}{\phi^6(q)} |\beta|^{-2-2/k} d\beta \right) \\ = O(P^{k+2} L^{2-6k}) = O(P^{k+2} L^{-7}). \end{aligned}$$

LEMMA 11.

$$\int_0^1 |V_2^2(\alpha) V_k(\alpha) - \sum_{q \leq L'} \sum_{\substack{a=1 \\ (a,q)=1}}^q V_2^{*2}(\alpha, a, q) V_k^*(\alpha, a, q)|^2 d\alpha = O(P^{k+2} L^{-7}).$$

Proof. By Lemmas 8, 9, and 10.

LEMMA 12.

$$\sum_{q \leq L'} \sum_{\substack{a=1 \\ (a,q)=1}}^q V_2^{*2}(\alpha, a, q) V_k^*(\alpha, a, q) = \sum_n \mathfrak{S}(n) \psi(n) e(n\alpha),$$

where, for $6 \leq n \leq N$,

$$c_7 n^{1/k} / \log^3 n < \psi(n) < c_8 n_1^{1/k} \log^3 n.$$

Proof. The sum on the left is

$$\begin{aligned} \sum_{q \leq L'} \sum_{\substack{a=1 \\ (a,q)=1}}^q \frac{(S_{a,q}^{(2)})^2 S_{a,q}^{(k)}}{\psi^3(q)} \sum_{n_1=2}^N \frac{1}{n_1^{\frac{1}{2}} \log n_1} \times \\ \times \sum_{n_2=1}^N \frac{1}{n_2^{\frac{1}{2}} \log n_2} \sum_{n_3=1}^N \frac{1}{n_3^{1-1/k} \log n_3} e(n_1 + n_2 + n_3) \left(\alpha - \frac{a}{q} \right) \\ = \sum_n \mathfrak{S}(n) \psi(n) e(n\alpha), \end{aligned}$$

where

$$\psi(n) = \sum_{\substack{n_1=2 \\ n_1+n_2+n_3=n}}^N \sum_{\substack{n_2=2 \\ n_1+n_2+n_3=n}}^N \sum_{\substack{n_3=2 \\ n_1+n_2+n_3=n}}^N \frac{1}{n_1^{\frac{1}{2}} \log n_1 \cdot n_2^{\frac{1}{2}} \log n_2 \cdot n_3^{1-1/k} \log n_3}.$$

For $6 \leq n \leq N$, $\psi(n)$ evidently satisfies the inequality.

LEMMA 13. Let p not be one of 2, 3, 5. If $p \equiv 3 \pmod{4}$, $(p-1) \mid k$, and $n \equiv 1 \pmod{p}$, then

$$1 + A_p(n) = 0;$$

otherwise

$$1 + A_p(n) > 0.$$

Proof. Let M be the number of solutions of

$$x^2 + y^2 + z^k \equiv n \pmod{p}, \quad p \nmid xyz. \quad (1)$$

Then

$$\begin{aligned} 1 + A_p(n) &= 1 + \frac{1}{(p-1)^3} \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} \sum_{z=1}^{p-1} e_p\{a(x^2 + y^2 + z^k - n)\} \\ &= 1 + \frac{1}{(p-1)^3} \{(p-1)M - (p-1)^3 + M\} = \frac{p}{(p-1)^3} M. \end{aligned}$$

Thus it is sufficient to consider M .

(1) Suppose $(p-1) \nmid k$. Then z^k gives at least two different values as z runs through $1, \dots, p-1 \pmod{p}$. Therefore we can always choose z ($p \nmid z$) such that $n - z^k \not\equiv 0 \pmod{p}$.

(1.1) If $\left(\frac{n-z^k}{p}\right) = 1$, then the congruence (1) is equivalent to

$$X^2 + 1 \equiv Y^2 \pmod{p}, \quad p \nmid XY. \quad (2)$$

For $p > 5$ there always exists an integer c such that c^2 is not congruent to 1 or $-1 \pmod{p}$. Thus

$$X = \frac{1}{2}(c - c^{p-2}), \quad Y = \frac{1}{2}(c + c^{p-2})$$

satisfies (2). Hence $M \geq 1$.

(1.2) Let $\left(\frac{n-z^k}{p}\right) = -1$. If $x^2 + y^2 \equiv n - z^k \pmod{p}$ were not solvable, then the $\frac{1}{2}(p+1)$ incongruent numbers

$$n - z^k - y^2 \quad \{y = 0, \dots, \frac{1}{2}(p-1)\}$$

would be quadratic non-residues to modulus p , which is impossible. Therefore there are integers x and y such that

$$x^2 + y^2 + z^k \equiv n \pmod{p}.$$

Since $\left(\frac{n-z^k}{p}\right) = -1$, we have $p \nmid xy$. Thus $M \geq 1$.

(2) Suppose $(p-1) \mid k$. Then $z^k \equiv 1 \pmod{p}$. (1) is equivalent to

$$x^2 + y^2 \equiv n-1 \pmod{p}, \quad p \nmid xy. \quad (3)$$

(2.1) If $n \not\equiv 1 \pmod{p}$, the lemma can be proved in the same way as (1).

(2.2) Let $n \equiv 1 \pmod{p}$.

(2.21) Suppose $p \equiv 1 \pmod{4}$. Evidently there are x and y satisfying

$$x^2 + y^2 \equiv p \equiv n-1 \pmod{p}, \quad p \nmid xy.$$

(2.22) Suppose $p \equiv 3 \pmod{4}$. It is also evident that

$$x^2 + y^2 \equiv 0 \pmod{p}, \quad p \nmid xy$$

has no solution. That is, $M = 0$.

LEMMA 14.

$$1 + A_2(n) + A_{2^2}(n) + A_{2^3}(n) \begin{cases} = 0 & \text{if } 2 \mid k \text{ and } n \not\equiv 3 \pmod{8}, \\ = 0 & \text{if } 2 \nmid k \text{ and } 2 \mid n, \\ > 0 & \text{otherwise.} \end{cases}$$

Proof. By the same arguments as Lemma 13 we have

$$1 + A_2(n) + A_{2^2}(n) + A_{2^3}(n) = 2^3 M / \phi^3(2^3),$$

where M is the number of solutions of

$$x^2 + y^2 + z^k \equiv n \pmod{2^3}, \quad 2 \nmid xyz.$$

This congruence is equivalent to

$$z^k \equiv n-2 \pmod{2^3}.$$

(1) If k is even, then n must be congruent to 3 (mod 8), for otherwise the congruence has no solution.

(2) If k is odd, then the congruence is equivalent to

$$z \equiv n-2 \pmod{2^3}, \quad 2 \nmid z,$$

which is always solvable for odd n .

LEMMA 15.

$$1 + A_3(n) \begin{cases} = 0 & \text{if } 2 \mid k, 3 \nmid n, \\ = 0 & \text{if } 2 \nmid k, n \equiv 2 \pmod{3}, \\ > 0 & \text{otherwise.} \end{cases}$$

LEMMA 16.

$$1 + A_5(n) \begin{cases} = 0 & \text{if } 4 \mid k \text{ and } n \equiv 2 \text{ or } 0 \pmod{5}, \\ = 0 & \text{if } k \equiv 2 \pmod{4} \text{ and } n \equiv 0 \pmod{5}, \\ > 0 & \text{otherwise.} \end{cases}$$

These two lemmas can be proved in the same way as Lemma 14.

LEMMA 17.

$$A_p(n) = O(p^{-1}).$$

Therefore

$$A_q(n) = O(q^{-1+\epsilon}).$$

Proof. Let $p > 2$, and let M be the number of solutions of

$$y^k \equiv n \pmod{p}, \quad p \nmid y.$$

Then evidently $M \leq k$.

By well-known theorems on Gaussian sums we have

$$\begin{aligned} \phi^3(p)A_p(n) &= \sum_{a=1}^{p-1} \left\{ \sum_{x=1}^{p-1} e_p(ax^2) \right\}^2 \left\{ \sum_{y=1}^{p-1} e_p(ay^k) \right\} e_p(-an) \\ &= \sum_{y=1}^{p-1} \left\{ \sum_{a=1}^{p-1} \left[\left(\frac{a}{p} \right) i^{k(p-1)^2 \sqrt{p}-1} \right]^2 e_p[-a(n-y^k)] \right\} \\ &= \sum_{y=1}^{p-1} \left\{ [(-1)^{k(p-1)} p + 1] \sum_{a=1}^{p-1} e_p[-a(n-y^k)] - \right. \\ &\quad \left. - 2i^{k(p-1)^2 \sqrt{p}} \sum_{a=1}^{p-1} \left(\frac{a}{p} \right) e_p[-a(n-y^k)] \right\} \\ &= \{(-1)^{k(p-1)} p + 1\} \{pM - (p-1)\} - 2p \sum_{y=1}^{p-1} \left(\frac{n-y^k}{p} \right). \end{aligned}$$

Since $M \leq k$, we have

$$\phi^3(p)A_p(n) = O(p^2).$$

The second part of the lemma follows from Lemma 4, since

$$A_{q_1 q_2}(n) = A_{q_1}(n)A_{q_2}(n)$$

for $(q_1, q_2) = 1$.

LEMMA 18.

$$\sum_{n=1}^N \{\mathfrak{S}(n) - \mathfrak{S}_1(n)\}^2 = O(P^k L^{-1+\epsilon}).$$

Proof. By Lemma 4, $A_{p^t}(n) = 0$ for $t > 1$, $p \neq 2$, and $A_{2^t}(n) = 0$ for $t > 3$. Since

$$4 \prod_{p \leq \frac{1}{2}L} p < L^{\pi(\frac{1}{2}L)} < N^{\frac{1}{2}}$$

and $\frac{1}{3}L < L^{\frac{1}{2}} < N^{\frac{1}{2}}$, we have

$$\mathfrak{S}(n) - \mathfrak{S}_1(n) = \sum_{\frac{1}{2}L \leq q \leq N^{\frac{1}{2}}} \theta(q)A_q(n),$$

where $\theta(q)$ is 0, 1, or -1 , and is independent of n . Therefore

$$\begin{aligned} \sum_{n=1}^N \{\mathfrak{S}(n) - \mathfrak{S}_1(n)\}^2 &= \sum_{n=1}^N \sum_{\frac{1}{2}L \leq q \leq N^{\frac{1}{2}}} \theta^2(q)A_q^2(n) + 2 \sum_{\frac{1}{2}L \leq q_1 < q_2 \leq N^{\frac{1}{2}}} \theta(q_1)\theta(q_2) \sum_{n=1}^N A_{q_1}(n)A_{q_2}(n) \\ &= \sum_1 + \sum_2, \quad \text{say.} \end{aligned}$$

By Lemma 17,

$$\left| \sum_{\frac{1}{2}L \leq q \leq N^{\frac{1}{2}}} \theta^2(q) A_q^2(n) \right| = O\left(\sum_{q \geq \frac{1}{2}L} \frac{1}{q^{2-\epsilon}} \right) = O(L^{-1+\epsilon}).$$

Hence $\sum_1 = O(NL^{-1+\epsilon})$.

To deal with \sum_2 we observe that, if $q_1 \neq q_2$ and $(a_1, q_1) = (a_2, q_2) = 1$, then $a_1 q_2 + a_2 q_1 \not\equiv 0 \pmod{q_1 q_2}$. Hence $A_{q_1}(n) A_{q_2}(n)$ is a periodic function in n with period $q_1 q_2$ and average value zero. Hence

$$\left| \sum_{n=1}^N A_{q_1}(n) A_{q_2}(n) \right| \leq \sum_{n=1}^{q_1 q_2} |A_{q_1}(n) A_{q_2}(n)| = O(q_1^\epsilon q_2^\epsilon).$$

Thus $\sum_2 = O\left(\sum_{\frac{1}{2}L \leq q_1 < q_2 \leq N^{\frac{1}{2}}} q_1^\epsilon q_2^\epsilon \right) = O(N^{\frac{1}{2}+\epsilon}) = O(NL^{-1})$.

LEMMA 19. If n satisfies the conditions of Theorem 1, then

$$\mathfrak{S}_1(n) > c_9 (\log \log P)^{-c_{10}}.$$

Proof. By Lemmas 13, 14, 15, 16, 17,

$$\begin{aligned} \mathfrak{S}_1(n) &\geq \{1 + A_2(n) + A_4(n) + A_8(n)\} \prod_{2 < p \leq c_{11}} \{1 + A_p(n)\} \prod_{c_{11} < p < \frac{1}{2}L} \left(1 - \frac{c_{11}}{p}\right) \\ &> c_{12} \prod_{c_{11} < p < \frac{1}{2}L} \left(1 - \frac{c_{11}}{p}\right) \\ &> c_{13} \prod_{p < \frac{1}{2}L} (1 - p^{-1})^{c_{11}} \\ &> c_{14} (\log \log P)^{-c_{11}}. \end{aligned}$$

Proof of Theorem 1. By Lemmas 11, 12, 18,

$$\sum_{n=3}^N \{r(n) - \mathfrak{S}_1(n)\psi(n)\}^2 = O(P^{k+2} L^{-7+\epsilon}).$$

Thus, by Lemma 19,

$$\sum_{\substack{n=3 \\ r(n)=0}}^{N'} \frac{n^{2/k}}{\log^6 n} (\log \log P)^{-c_{10}} = O(P^{k+2} L^{-7+\epsilon}),$$

where the dash denotes that n is restricted to the integers considered in Theorem 1. And so

$$\sum_{\substack{n=3 \\ r(n)=0}}^{N'} n^{2/k} = O(P^{k+2} L^{-1+2\epsilon}).$$

Let M denote the number of integers n belonging to A_k for which $3 \leq n \leq N$ and $r(n) = 0$. Then

$$\frac{1}{1+2/k} M^{1+2/k} \leq \sum_{n=1}^M n^{2/k} \leq \sum_{\substack{n=3 \\ r(n)=0}}^{N'} n^{2/k} = O(P^{k+2} L^{-1+2\epsilon}).$$

Hence

$$M = O(NL^{-(1-2\epsilon)k/(k+2)}), \\ = o(N).$$

4. Proof of Theorem 2

$$\text{Let} \quad V_1(\alpha)V_2^2(\alpha)V_k(\alpha) = \sum_n r'(n)e(n\alpha),$$

so that, for $n \leq N$, $r'(n)$ is the number of representations of n as $p_1 + p_2^2 + p_3^2 + p_4^k$. Let

$$A'_q(N) = \frac{M(q)}{\phi^4(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^q (S_{a,q}^{(2)})^2 S_{a,q}^{(k)} e_q(-Na), \\ \mathfrak{S}'(N) = \sum_{q=1}^{\infty} A'_q(N).$$

For convenience we divide \mathfrak{M} into two parts

$$\mathfrak{M}_1 \quad |\beta| > P^{-k}L^{hk}, \\ \mathfrak{M}_2 \quad |\beta| \leq P^{-k}L^{hk}.$$

LEMMA 20.

$$\left(\sum_{\mathfrak{M}} \int + \sum_{\mathfrak{M}_1} \int \right) |V_1(\alpha)V_2^2(\alpha)V_k(\alpha)| d\alpha = O(P^{k+1}L^{-h}).$$

Proof. By Lemma 2, on \mathfrak{M} ,

$$V_k(\alpha) = O(PL^{-h}).$$

Further, by Lemmas 6 and 7, on \mathfrak{M}_1 ,

$$V_k(\alpha) = O(PL^{-h}).$$

Therefore, by Lemma 3,

$$\left(\sum_{\mathfrak{M}} \int + \sum_{\mathfrak{M}_1} \int \right) |V_1(\alpha)V_2^2(\alpha)V_k(\alpha)| d\alpha = O\left(PL^{-h} \int_0^1 |V_1(\alpha)V_2^2(\alpha)| d\alpha \right) \\ = O\left(PL^{-h} \sqrt{\int_0^1 |V_1(\alpha)|^2 d\alpha \int_0^1 |V_2(\alpha)|^4 d\alpha} \right) O(P^{k+1}L^{-h}).$$

LEMMA 21.

$$\sum_{\mathfrak{M}_2} \int |V_1(\alpha)V_2^2(\alpha)V_k(\alpha) - V_1^*(\alpha, a, q)V_2^{*2}(\alpha, a, q)V_k^*(\alpha, a, q)| d\alpha \\ = O(P^{k+1}e^{-c_{15}\sqrt{L}}).$$

The proof of the lemma is similar to that of Lemma 9.

LEMMA 22.

$$\sum_{\mathfrak{M}} \left(\int_0^1 - \int_{\mathfrak{M}_2} \right) |V_1^*(\alpha, a, q)V_2^{*2}(\alpha, a, q)V_k^*(\alpha, a, q)| d\alpha = O(P^{k+1}L^{-h}).$$

The proof of the lemma is easier than that of Lemma 10.

LEMMA 23.

$$r'(N) = \mathfrak{S}'(N)\psi'(N) + O(P^{k+1}L^{-h}),$$

where

$$\psi'(N) = \sum_{\substack{n_1=2 \\ n_1+n_2+n_3+n_4=N}}^N \sum_{n_2=2}^N \sum_{n_3=2}^N \sum_{n_4=2}^N \frac{1}{\log n_1 \cdot n_2^{\frac{1}{2}} \log n_2 \cdot n_3^{\frac{1}{2}} \log n_3 \cdot n_4^{1-1/k} \log n_4}.$$

Obviously $\psi'(N)$ satisfies the inequality

$$\frac{c_{16} N^{1+1/k}}{\log^4 N} < \psi'(N) < \frac{c_{17} N^{1+1/k}}{\log^4 N}.$$

Proof. By Lemmas 20, 21, 22 we obtain

$$\begin{aligned} r'(N) &= \int_0^1 V_1(\alpha) V_2^*(\alpha) V_k(\alpha) e(-N\alpha) d\alpha \\ &= \sum_{\mathfrak{M}} \int_0^1 V_1^*(\alpha, a, q) V_2^{*2}(\alpha, a, q) V_k^*(\alpha, a, q) e(-N\alpha) d\beta + O(P^{k+1}L^{-h}) \\ &= \sum_{q \leq L} A'_q(N) \int_0^1 \sum_{n_1=2}^N \sum_{n_2=2}^N \sum_{n_3=2}^N \sum_{n_4=2}^N \frac{e\{(n_1+n_2+n_3+n_4-N)\beta\} d\beta}{\log n_1 \cdot n_2^{\frac{1}{2}} \log n_2 \cdot n_3^{\frac{1}{2}} \log n_3 \cdot n_4^{1-1/k} \log n_4} + \\ &\quad + O(P^{k+1}L^{-h}) \\ &= \sum_{q \leq L^{h_0}} A'_q(N) \psi'(N) + O(P^{k+1}L^{-h}). \end{aligned}$$

The lemma follows since ($h < h_0$)

$$\sum_{q > L^{h_0}} A'_q(N) = O\left(\sum_{q > L^{h_0}} q^{-1+\epsilon}\right) = O(L^{-h_0+\epsilon}).$$

LEMMA 24. If N is even and prime to 3 for even k , then

$$\mathfrak{S}'(N) \geq c_{18} > 0.$$

Proof. It is easy to obtain

$$\mathfrak{S}'(N) = \prod_p \{1 + A'_p(N)\},$$

and we have

$$\begin{aligned} A'_p(N) &= -\frac{1}{\phi(p)} A_p(N) \\ &= -\frac{1}{\phi(p)} \left(\frac{pM}{(p-1)^3} - 1 \right) \end{aligned}$$

(cf. Lemma 13), where M is the number of solutions of the congruence

$$x^2 + y^2 + z^k \equiv N \pmod{p}, \quad p \nmid xyz.$$

Thus

$$1 + A'_p(N) = 1 + \frac{1}{\phi(p)} - \frac{pM}{\phi^4(p)} = 0$$

if and only if $M = \phi^3(p)$. The only cases for which $M = \phi^3(p)$ are (i) $p = 2$, $N \equiv 1 \pmod{2}$, and (ii) $p = 3$, $N \equiv 0 \pmod{3}$ for even k . Therefore, in the case considered,

$$1 + A'_p(N) > 0.$$

On the other hand, by Lemma 17,

$$A'_p(N) = O(p^{-2}).$$

We have thus

$$\mathfrak{S}'(N) > c_{19} \prod_{p > c_{19}} (1 - c_{20} p^{-2}) \geq c_{18} > 0.$$

Theorem 2 is therefore proved by Lemmas 23 and 24.

In particular, for $k = 2$, the singular series can be evaluated by Gaussian sums: namely, for even N ,

$$\mathfrak{S}'(N) = 2 \prod_{p > 2} \left[1 + r(p) \left\{ p^2 \left(\frac{-N}{p} \right) + 3p \left[\left(\frac{N}{p} \right) + \left(\frac{-1}{p} \right) \right] + 1 \right\} \right],$$

$$\text{where } r(p) = \begin{cases} -\frac{1}{(p-1)^4} & \text{for } p \nmid N, \\ \frac{1}{(p-1)^3} & \text{for } p \mid N, \end{cases}$$

and (N/p) is Kronecker's symbol.

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